CONCERNING THE DEFINITION OF HARMONIC FUNCTIONS

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1. Introduction. A real function \( u(x, y) \), defined in a domain (non-null connected open set) \( D \), is said to be harmonic in \( D \) provided \( u(x, y) \) and its partial derivatives of the first and second orders are continuous and the Laplace equation,

\[
\Delta u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0,
\]

is satisfied throughout \( D \). A function is said to be harmonic at a point provided it is harmonic in a domain containing the point.

It has been shown \cite{1} that if \( u(x, y) \) is continuous in \( D \) and if the second order partial derivatives \( \frac{\partial^2 u}{\partial x^2} \) and \( \frac{\partial^2 u}{\partial y^2} \) exist and satisfy the Laplace equation (1) throughout \( D \), then \( u(x, y) \) is harmonic in \( D \).

We shall show that if \( u(x, y) \) and its partial derivatives \( \frac{\partial u}{\partial x} \) and \( \frac{\partial u}{\partial y} \) are continuous in \( D \), if \( \frac{\partial u}{\partial x} \) and \( \frac{\partial u}{\partial y} \) are differentiable, or even have finite Dini derivatives, with respect to \( x \) and \( y \) at all points of \( D \) except at most at the points of a denumerable set of points in \( D \), and if the Laplace equation (1) is satisfied at almost all points of \( D \) at which \( \frac{\partial^2 u}{\partial x^2} \) and \( \frac{\partial^2 u}{\partial y^2} \) exist, then \( u(x, y) \) is harmonic in \( D \).

Our result is comparable with the Looman-Menchoff theorem \cite[pp. 9–16; 5, pp. 198–201]{3} concerning the Cauchy-Riemann first order partial differential equations and analytic functions of a complex variable. Ridder \cite{4} has stated that harmonic functions can be given a Looman-Menchoff characterization; but a generalization of the Looman-Menchoff theorem on which his proof is based is invalid, for there are functions having isolated singularities which satisfy the hypotheses of the generalization without satisfying the conclusion. For a generalization of the Looman-Menchoff theorem, see Maker \cite{2}.

2. Notation and lemmas. By \( C(Q) \) we shall denote a square, by \( C(R) \) a rectangle, having sides parallel to the coordinate axes. The set consisting of the points of \( C(Q) \), or of \( C(R) \), plus its interior, will be denoted by \( Q \), or \( R \), respectively.

Let \( F \) be a non-null set closed with respect to the domain \( D \), and \( C(Q) \) any square with \( Q \) lying in \( D \), with sides of positive length and parallel to the coordinate axes, and with center at a point of \( F \). Then the points common to \( F \) and \( Q \) will be called a portion of \( F \).

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\(^1\) Numbers in brackets refer to the references cited at the end of the paper.
We shall use as lemmas the following known results.

**Lemma 1.** If \( u(x, y) \) is continuous at \((x_0, y_0)\) and harmonic in a deleted neighborhood of \((x_0, y_0)\), then \( u(x, y) \) is harmonic at \((x_0, y_0)\).

**Proof.** This follows from the fact that the function can be expanded in a two-way power series in a deleted neighborhood of \((x_0, y_0)\).

**Lemma 2.** If \( u(x, y) \) is harmonic in the finite domain \( D \), then for each rectangle \( C(R) \) such that \( R \) lies in \( D \) we have

\[
\int_{C(R)} \frac{du}{dv} ds = 0,
\]

where \( d/dv \) denotes differentiation in the direction of the outward normal.

**Proof.** The result follows directly from Green’s theorem,

\[
\int_{C(R)} \frac{du}{dv} ds = \iint_{R} \left( \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} \right) dxdy.
\]

**Lemma 3.** If \( u(x, y) \) and its partial derivatives of the first order are continuous in \( D \) and if for every square \( C(Q) \) having sides parallel to the coordinate axes and such that \( Q \) lies in \( D \) we have

\[
\int_{C(Q)} \frac{du}{dv} ds = 0,
\]

then \( u(x, y) \) is harmonic in \( D \).

**Proof.** The mean-value function,

\[
u^{(r)}(x, y) \equiv \frac{1}{4r^2} \int_{-r}^{r} \int_{-r}^{r} u(x + \xi, y + \eta) d\xi d\eta,
\]

has continuous partial derivatives of the second order in the part \( D^{(r)} \) of \( D \) in which \( u^{(r)}(x, y) \) is defined. We have

\[
\frac{\partial^2 u^{(r)}(x, y)}{\partial x^2} = \frac{1}{4r^2} \int_{-r}^{r} \left[ \frac{\partial u(x + \xi, y + \eta)}{\partial x} - \frac{\partial u(x - \xi, y + \eta)}{\partial x} \right] d\eta
\]

and

\[
\frac{\partial^2 u^{(r)}(x, y)}{\partial y^2} = \frac{1}{4r^2} \int_{-r}^{r} \left[ \frac{\partial u(x + \xi, y + \eta)}{\partial y} - \frac{\partial u(x + \xi, y - \eta)}{\partial y} \right] d\xi,
\]

whence
If \( du(r(x, y)) = I - ds \),
where \( C(Q_r) \) is the square with sides of length \( 2r \) and parallel to the coordinate axes, and with center at \((x, y)\).

From (3) and (4) it follows that \( u(r(x, y)) \) is harmonic in \( D^r \). Hence \( u(x, y) \), the uniform limit of \( u(r(x, y)) \) on any closed and bounded subset of \( D \) as \( r \to 0 \), is harmonic in \( D \).

**LEMMA 4 (BAIRE’S THEOREM).** Let \( F \) be a non-null plane set lying in a domain \( D \) and closed with respect to \( D \), and let \( \{ F_n \}, n = 1, 2, \ldots \), be a sequence of sets lying in \( D \) and closed with respect to \( D \) such that \( \{ F_n \} \) covers \( F \); that is, each point of \( F \) is a point of at least one \( F_n \). Then there is a member \( F_N \) of \( \{ F_n \} \) which contains a portion of \( F \).

**PROOF.** If the result were not valid, there would be a descending sequence \( \{ P_n \}, P_1 \supset P_2 \supset \cdots \), of portions of \( F \) such that \( P_n \) and \( F_n \) have no point in common, \( n = 1, 2, \ldots \). Then \( \bigcap_{n=1}^\infty P_n \) and \( \sum_{n=1}^\infty F_n \) would have no point in common. But this is impossible since \( \bigcap_{n=1}^\infty P_n \) contains a point of \( F \) and \( \sum_{n=1}^\infty F_n \) covers \( F \).

**LEMMA 5.** Let \( C(Q) \) be a square having sides parallel to the coordinate axes, let \( F \) be a closed non-null set in \( Q \), let \( C(R) \) be the smallest rectangle (which may be degenerate) having sides parallel to the coordinate axes and satisfying the condition that \( F \) is contained in \( R \), and let the vertices of \( C(R) \) have coordinates

\[
(x_1, y_1), (x_2, y_1), (x_2, y_2), (x_1, y_2), \quad x_1 \leq x_2, y_1 \leq y_2.
\]

If the real function \( w(x, y) \) is defined on the set \( Q \), if the first order partial derivatives of \( w(x, y) \) exist, or even if the Dini derivatives are finite, at every point of \( Q \) except at most at the points of a denumerable set of points in \( Q \), and if for the finite constant \( N \) we have

\[
| w(x_0 + h, y_0) - w(x_0, y_0) | \leq N | h |,
\]

\[
| w(x_0, y_0 + k) - w(x_0, y_0) | \leq N | k |,
\]

for all \((x_0, y_0)\) in \( F \) and all \((x_0 + h, y_0), (x_0, y_0 + k)\) in \( Q \), then

\[
\left| \int_{x_1}^{x_2} [w(x, y_2) - w(x, y_1)] dx - \int_{x_1}^{x_2} \int_{y_1}^{y_2} \frac{\partial w}{\partial y} dxdy \right| \leq 5N \text{ meas } (Q - F),
\]

\[
\left| \int_{y_1}^{y_2} [w(x_2, y) - w(x_1, y)] dy - \int_{y_1}^{y_2} \int_{x_1}^{x_2} \frac{\partial w}{\partial x} dxdy \right| \leq 5N \text{ meas } (Q - F).
\]

**PROOF.** A proof of Lemma 5 may be found in Saks [5, pp. 198–199] or Menchoff [3, pp. 10–12].
We note, relative to Lemma 5, that since \( w(x, y) \) is differentiable, or has finite Dini derivatives, with respect to \( x \) and \( y \) at all points of \( Q \) except at most at the points of a denumerable set of points in \( Q \), it follows [5, pp. 236, 272] that \( \partial w/\partial x \) and \( \partial w/\partial y \) exist almost everywhere in \( Q \) and are integrable.

3. Theorem. We shall establish the following result.

**Theorem.** If the real function \( u(x, y) \) and its first order partial derivatives with respect to \( x \) and \( y \) are continuous in the finite domain \( D \), and if \( \partial u/\partial x \) and \( \partial u/\partial y \) are differentiable, or even have finite Dini derivatives, with respect to \( x \) and \( y \) at all points of \( D \) except at most at the points of a denumerable set of points in \( D \), and if the Laplace equation,

\[
\Delta u = \partial^2 u/\partial x^2 + \partial^2 u/\partial y^2 = 0,
\]

is satisfied at almost all points at which \( \partial^2 u/\partial x^2 \) and \( \partial^2 u/\partial y^2 \) exist in \( D \), then \( u(x, y) \) is harmonic in \( D \).

**Proof.** Suppose that there is a point of \( D \) at which \( u(x, y) \) is not harmonic; we shall obtain a contradiction.

Denote the set of points of \( D \) at which \( u(x, y) \) is not harmonic by \( F \). Since by definition the set of points at which \( u(x, y) \) is harmonic is open, it follows that \( F \) is closed with respect to \( D \). Further, by Lemma 1, \( F \) has no isolated points. Hence \( F \) is perfect with respect to \( D \).

For each positive integer \( n \), let \( F_n \) be the set of points \((x, y)\) of \( F \) for which

\[
\left| \frac{\partial u(x+h, y)}{\partial x} - \frac{\partial u(x, y)}{\partial x} \right| \leq n |h|,
\]

\[
\left| \frac{\partial u(x, y+h)}{\partial x} - \frac{\partial u(x, y)}{\partial x} \right| \leq n |h|,
\]

\[
\left| \frac{\partial u(x+h, y)}{\partial y} - \frac{\partial u(x, y)}{\partial y} \right| \leq n |h|,
\]

\[
\left| \frac{\partial u(x, y+h)}{\partial y} - \frac{\partial u(x, y)}{\partial y} \right| \leq n |h|,
\]

for all \( h \) satisfying \( |h| \leq 1/n \). Since \( \partial u/\partial x \) and \( \partial u/\partial y \) are continuous in \( D \), it follows that the sets \( F_n \) are closed with respect to \( D \); and since \( \partial u/\partial x \) and \( \partial u/\partial y \) are differentiable, or have finite Dini derivatives, with respect to \( x \) and \( y \) at all points of \( D \) except at most at the points of a denumerable set of points in \( D \), it follows that the sets \( \{ F_n \} \),

Theorem.

1. Define.

2. Theorem.

3. Theorem.
n = 1, 2, \cdots, cover all of F except the points of a set H which is at most denumerable.

Then we have

(5) \[ F = \sum_{n=1}^{\infty} F_n + H. \]

It follows from (5) and Lemma 4 that there is a portion P of F either consisting of a single isolated point of H, or contained in an FN of \{F_n\}. But since F is perfect with respect to D, the former alternative is impossible, so that the latter alternative holds.

The above portion P of F is contained in F_N, and is the common part of F and a set Q_0 in D, where C(Q_0) is a square in D with center at a point of F and with sides parallel to the coordinate axes.

Let C(Q) be any square lying in Q_0 and having its sides parallel to the coordinate axes, and let F \cdot Q be the common part of F and Q. Let the sides of C(Q) be divided into n equal parts, with n so large that the length of each part is less than or equal to 1/N. Lines through the points of division parallel to the coordinate axes divide Q into n^2 squares. Let Q_{p,n}, p = 1, 2, \cdots, l; l \leq n^2, denote those of the n^2 squares having points in common with F.

For each Q_{p,n} let C(R_{p,n}) be the smallest rectangle (which may be degenerate) having sides parallel to the coordinate axes and such that R_{p,n} contains F \cdot Q_{p,n}. Let the vertices of C(R_{p,n}) have coordinates

\[(x_{1,p,n}, y_{1,p,n}), (x_{2,p,n}, y_{1,p,n}), (x_{2,p,n}, y_{2,p,n}), (x_{1,p,n}, y_{2,p,n}), \quad x_{1,p,n} \leq x_{2,p,n}, y_{1,p,n} \leq y_{2,p,n}.\]

By Lemma 2 and the uniform continuity of u(x, y) in Q, we have

(6) \[ \int_{C(Q)} \frac{du}{dv} \, ds = \sum_{p=1}^{l} \int_{C(R_{p,n})} \frac{du}{dv} \, ds. \]

Since

\[ \int_{C(R_{p,n})} \frac{du}{dv} \, ds = \int_{x_{1,p,n}}^{x_{2,p,n}} \left[ \frac{\partial u(x, y_{2,p,n})}{\partial y} - \frac{\partial u(x, y_{1,p,n})}{\partial y} \right] dx + \int_{y_{1,p,n}}^{y_{2,p,n}} \left[ \frac{\partial u(x_{2,p,n}, y)}{\partial x} - \frac{\partial u(x_{1,p,n}, y)}{\partial x} \right] dy, \]

by Lemma 5 we have

(7) \[ \left| \int_{C(R_{p,n})} \frac{du}{dv} \, ds - \int_{F \cdot Q_{p,n}} \left( \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} \right) dx dy \right| \leq 10N \text{ meas } (Q_{p,n} - F \cdot Q_{p,n}). \]
From (6) and (7) we obtain

\[
\left| \int_{C(Q)} \frac{du}{dv} \, ds - \int_{F \cdot Q} \left( \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} \right) \, dx \, dy \right| 
\leq 10N \text{ meas} \left( \sum_{p=1}^{l} Q_{p,n} - F \cdot Q \right). 
\]

Since the lengths of the sides of the squares \( C(Q_{p,n}) \to 0 \) as \( n \to \infty \), it follows that

\[
\lim_{n \to \infty} \text{meas} \left( \sum_{p=1}^{l} Q_{p,n} - F \cdot Q \right) = 0. 
\]

By hypothesis and the note at the end of §2, (1) is satisfied almost everywhere in \( D \). Consequently from (8) and (9) we obtain

\[
\int_{C(Q)} \frac{du}{dv} \, ds = 0. 
\]

Hence by Lemma 3, \( u(x, y) \) is harmonic in \( Q_0 \). But the center of \( Q_0 \) is a point of \( R \), so that \( u(x, y) \) is not harmonic in \( Q_0 \). Thus the supposition that there is a point of \( D \) at which \( u(x, y) \) is not harmonic has led to a contradiction.

REFERENCES

2. P. T. Maker, Conditions on \( u(x, y) \) and \( v(x, y) \) necessary and sufficient for the regularity of \( u+iv \), Trans. Amer. Math. Soc. vol. 45 (1939) pp. 265–275.