A RELATION BETWEEN JACOBI AND
LAGUERRE POLYNOMIALS

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1. Introduction. We shall obtain an apparently new expression for
the Jacobi polynomial \( P^{(\alpha, \beta)}_n(x) \) as a series of products of Laguerre
polynomials of orders \( \alpha \) and \( \beta \) and of different arguments.

Because of the popularity of the Legendre and the simple Laguerre
polynomials, we shall prove that special case of the relation between
Jacobi and Laguerre polynomials. The method of proof carries
through with no alteration in the general case and need not be re­
peated.

The generating function

\[
e^{tJ_0(2(\sqrt{y})^{1/2})} = \sum_{n=0}^{\infty} L_n(y) \frac{t^n}{n!}
\]

for the Laguerre polynomials may be found in the standard books.
For the Legendre polynomials a special case of a well known result\(^1\)
of Bateman is

\[
I_0((2t(x - 1))^{1/2})I_0((2t(x + 1))^{1/2}) = \sum_{n=0}^{\infty} P_n(x) \frac{t^n}{(n!)^2}.
\]

From (1) with \( y = (1-x)/2 \) it follows, since \( J_0(\sqrt{y}) = I_0(y) \), that

\[
e^{iJ_0((2t(x - 1))^{1/2})} = \sum_{n=0}^{\infty} L_n \left(\frac{1-x}{2}\right) \frac{t^n}{n!}.
\]

Using (1) with \( y = (1+x)/2 \) and with \( t \) replaced by \( (-t) \), we may
conclude that

\[
e^{-tI_0((2t(x + 1))^{1/2})} = \sum_{n=0}^{\infty} (-1)^n L_n \left(\frac{1+x}{2}\right) \frac{t^n}{n!}.
\]

Then, with the aid of the Cauchy product of two series, it may be
seen that

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\[ I_0((2t(x - 1))^{1/2})I_0((2t(x + 1))^{1/2}) \]
\[ = \sum_{n=0}^\infty \sum_{k=0}^n (-1)^k C_{n,k} L_k \left( \frac{1 + x}{2} \right) L_{n-k} \left( \frac{1 - x}{2} \right) \frac{t^n}{n!}, \]

where \( C_{n,k} \) is the binomial coefficient.

Therefore, in view of (2), we have the desired result,

\[ P_n(x) = n! \sum_{k=0}^n (-1)^k C_{n,k} L_k \left( \frac{1 + x}{2} \right) L_{n-k} \left( \frac{1 - x}{2} \right). \]

3. A similar formula for the Jacobi polynomial. From Bateman\(^2\) again we may obtain the following generating function for the Jacobi polynomial:

\[ [2^{\alpha+\beta}t^{\alpha-\beta}(x - 1)^{-\alpha}(x + 1)^{-\beta}]^{1/2}I_\alpha((2t(x - 1))^{1/2})I_\beta((2t(x + 1))^{1/2}) \]
\[ = \sum_{n=0}^\infty P_n^{(\alpha,\beta)}(x) \frac{t^n}{\Gamma(n + \alpha + 1)\Gamma(n + \beta + 1)}. \]

For the Laguerre polynomial of order \( \alpha \) it is known that

\[ e^{t(yt^{1/2})}I_\alpha(2(yt^{1/2})) = \sum_{n=0}^\infty L_n^{(\alpha)}(y) \frac{t^n}{\Gamma(n + \alpha + 1)}. \]

These two results may be used just as we used (1) and (2) and they lead us to

\[ P_n^{(\alpha,\beta)}(x) = \sum_{k=0}^n (-1)^k \frac{\Gamma(n + \beta + 1)\Gamma(n + \alpha + 1)}{\Gamma(k + \beta + 1)\Gamma(n - k + \alpha + 1)} \]
\[ \cdot L_k^{(\beta)} \left( \frac{1 + x}{2} \right) L_{n-k}^{(\alpha)} \left( \frac{1 - x}{2} \right). \]

\(^2\) Bateman, loc. cit.