ON THE DEGREE OF APPROXIMATION OF FUNCTIONS
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1. Continuous functions. It has been proved by S. Bernstein that if \( f(x) \) is periodic and of the class Lip \( \alpha \), \( 0 < \alpha < 1 \), then the \((C, 1)\) means \( \sigma_n(x) = \sigma_n(x; f) \) of the Fourier series of \( f \) satisfy the condition

\[
\sigma_n(x) - f(x) = O(n^{-\alpha}),
\]

uniformly in \( x \). The result is false for \( \alpha = 1 \). The place of (1.1) is then taken by

\[
\sigma_n(x) - f(x) = O(\log n/n),
\]

and, as simple examples show, the factor \( \log n \) on the right cannot be removed (see, for example, A. Zygmund, Trigonometrical series, p. 62).

It will be shown here that for power series the inequality (1.1) holds even for \( \alpha = 1 \). More generally, we have the following theorem.

**Theorem 1.** Suppose that \( f(x) \) is periodic, continuous, and that the Fourier series of \( f \) is of power series type,

\[
f(x) \sim \sum_{\nu=0}^{\infty} c_\nu e^{i\nu x}.
\]

Then

\[
| \sigma_{n-1}(x) - f(x) | \leq A\omega(2\pi/n),
\]

where \( \omega(\delta) \) is the modulus of continuity of \( f \) and \( A \) is an absolute constant.

The proof is based on the following lemma.

**Lemma.** Suppose that

\[
g(x) \sim \sum_{\nu=0}^{+\infty} \gamma_\nu e^{i\nu x}
\]

satisfies \( |g(x+h) - g(x)| \leq M|h| \). Then

\[
|\tilde{\sigma}_{n-1}(x) - \tilde{g}(x)| \leq BM/n,
\]

where \( \tilde{g}(x) \) is the function conjugate to \( g(x) \) and \( \tilde{\sigma}_n(x) \) are the \((C, 1)\) means of the series conjugate to (1.4).

For the proof of the lemma we note that

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\[ \hat{g}(x) = -\frac{1}{\pi} \int_0^\pi [g(x + t) - g(x - t)] \frac{1}{2} \cot \frac{1}{2} t dt, \]

\[ \tau_{n-1}(x) = -\frac{1}{\pi} \int_0^\pi [g(x + t) - g(x - t)] \cdot \left[ \frac{1}{2} \cot \frac{1}{2} t - \frac{\sin nt}{n(2 \sin (t/2))^2} \right] dt, \]

\[ \hat{g}(x) - \tau_{n-1}(x) = \frac{1}{\pi} \int_0^\pi [g(x + t) - g(x - t)] \frac{\sin nt}{n(2 \sin (t/2))^2} dt \]

\[ = \frac{1}{\pi} \int_0^\pi \frac{\sin nu}{(2 \sin (u/2))^2} du, \]

say. Since \( |\sin nt| \leq n \sin t \leq n(2 \sin (t/2)) \) for \( 0 \leq t \leq \pi \),

\[ |P_n| \leq \frac{1}{\pi} \int_0^\pi \frac{2Mt}{2 \sin (t/2)} dt \leq \frac{M}{\pi} \int_0^\pi \frac{tdt}{(2/\pi)t/2} = \frac{M\pi}{n}. \]

In order to estimate \( Q_n \), we introduce the function

\[ \Lambda_n(t) = \frac{1}{\pi n} \int_t^\pi \frac{\sin nu}{(2 \sin (u/2))^2} du, \]

and integrate by parts. By the second mean value theorem,

\[ |\Lambda_n(t)| \leq \frac{2}{\pi n^2} \cdot \frac{1}{(2 \sin (t/2))^2} < \frac{\pi}{2n^2t^2}. \]

The function \( g \) is absolutely continuous and \( |g'(x)| \leq M \) almost everywhere. Thus

\[ |Q_n| \leq \frac{1}{\pi} \left| \int_0^\mu [g(x + t) - g(x - t)] \Lambda_n(t)dt \right| \]

\[ + \frac{1}{\pi} \left| \int_{\pi/n}^\pi [g'(x + t) + g'(x - t)] \Lambda_n(t)dt \right| \]

\[ \leq \frac{1}{\pi} \cdot \frac{2M}{n} \cdot \frac{\pi}{2n^2(\pi/n)^2} + \frac{2M}{\pi} \int_{\pi/n}^\pi |\Lambda_n(t)| dt \]

\[ < \frac{M}{\pi n} + \frac{M}{n^2} \int_{\pi/n}^\infty \frac{dt}{t^2} = \frac{2}{\pi} \frac{M}{n}. \]

This completes the proof of the lemma, with \( B = \pi + 2/\pi \).

Suppose now that the Fourier series of \( f \) is of power series type so
that \( \tilde{f} = -if \). If \(|f(x+h) - f(x)| \leq M\ |h|\), then

\[
|\sigma_{n-1}(x) - f(x)| = |\check{\sigma}_{n-1}(x) - \tilde{f}(x)| \leq BM/n. \tag{1.6}
\]

To complete the proof of Theorem 1, we introduce the function

\[
f_h(x) = \frac{1}{2h} \int_{-h}^{h} f(x + t) dt = [F(x + h) - F(x - h)]/2h
\]

\[
\sim \sum_{\nu=0}^{\infty} c_{\nu} e^{i\nu x} \left( \frac{\sin \nu h}{\nu h} \right),
\]

where \( F(x) \) is a primitive of \( f \). Hence \( df_h/dx \) exists, is continuous, and does not exceed \( \omega(2h)/2h \leq \omega(h)/h \) in absolute value. Moreover, the Fourier series of \( f_h \) is also of power series type. Now,

\[
|\sigma_{n-1}(x; f) - f(x)|
\]

\[
\leq |\sigma_{n-1}(x; f) - \sigma_{n-1}(x; f_h)| + |\sigma_{n-1}(x; f_h) - f_h(x)| + |f_h(x) - f(x)|
\]

\[
= \alpha_n + \beta_n + \gamma_n,
\]
say, and

\[
\gamma_n = \left| \frac{1}{2h} \int_{-h}^{h} [f(x + t) - f(x)] dt \right| \leq \omega(h),
\]

\[
\beta_n \leq B \frac{\omega(h)}{h} \frac{1}{n} \quad \text{(by (1.6)),}
\]

\[
\alpha_n = |\sigma_{n-1}(x; f - f_h)| \leq \max_x |f - f_h| \leq \omega(h).
\]

If we set \( h = 2\pi/n \) and collect the results, we obtain (1.3) with \( A = 2 + B/2\pi < 4 \).

2. Additional remarks. The foregoing proof of the lemma has certain disadvantages. First of all, it uses the result that a Lipschitz function is an indefinite integral, a fact which lies deeper than the assumptions of the lemma. Moreover, the argument does not work with the \( L^p \) metric. These difficulties are avoided by the following somewhat longer variant of the proof of the lemma. For the sake of brevity we do not compute the absolute constants \( C \) that occur in the proof.

Let \( P_n \) and \( Q_n \) have the same meaning as before, and let \( \psi(x, t) = f(x+t) - f(x-t) \). Hence

\[
|P_n| \leq \left| \frac{1}{\pi} \int_{0}^{\pi/n} \psi(x, t) \frac{\sin nt}{n(2 \sin (t/2))^2} dt \right| \leq \int_{0}^{\pi/n} |\psi(x, t)| t^{-1} dt.
\]

Let \( R_n(t) = 1/\pi n(2 \sin (t/2))^2 < 1/nt^2 \). Then, for \( n \geq 1 \),
\[ Q_n = \int_{x/n}^{x} \psi(x, t) R_n(t) \sin nt dt \]
\[ = -\int_{0}^{x/n} \psi(x, t + \pi/n) R_n(t + \pi/n) \sin nt dt, \]
\[ 2Q_n = \int_{x/n}^{x} \psi(x, t) [R_n(t) - R_n(t + \pi/n)] \sin nt dt \]
\[ + \int_{x/n}^{x-(n-1)/n} [\psi(x, t) - \psi(x, t + \pi/n)] R_n(t + \pi/n) \sin nt dt \]
\[ - \int_{0}^{x/n} \psi(x, t + \pi/n) R_n(t + \pi/n) \sin nt dt \]
\[ + \int_{x-(n-1)/n}^{x} \psi(x, t) R_n(t) \sin nt dt = I_n + J_n + K_n + L_n, \]
say.

By the mean-value theorem
\[ |R_n(t) - R_n(t + \pi/n)| \leq Cn^{-2}t^{-2}, \]
so that
\[ |I_n| \leq Cn^{-2} \int_{x/n}^{x-x/n} |\psi(x, t)| t^{-2} dt \leq Cn^{-2} \int_{x/n}^{x} |\psi(x, t)| t^{-2} dt. \]

Since \( R_n(t + \pi/n) \leq 1/nt^2 \), and
\[ \psi(x, t) - \psi(x, t + \pi/n) = \psi(x + t - \pi/2n, \pi/2n) - \psi(x - t - \pi/2n, \pi/2n), \]
we find
\[ |J_n| \leq Cn^{-1} \int_{x/n}^{x} |\psi(x + t - \pi/2n, \pi/2n)| t^{-2} dt \]
\[ + Cn^{-1} \int_{x/n}^{x} |\psi(x - t - \pi/2n, \pi/2n)| t^{-2} dt. \]

Moreover, since \( R_n(t + \pi/n) < Cn \) for \( 0 \leq t \leq \pi/n \),
\[ |K_n| \leq Cn \int_{0}^{\pi/n} |\psi(x, t + \pi/n)| dt. \]

Finally,
\[ |L_n| \leq Cn^{-1} \int_{x-(n-1)/n}^{x} |\psi(x, t)| dt = Cn^{-1} \int_{0}^{\pi/n} |\psi(x + \pi, t)| dt. \]
By assumption, $|\psi(x, u)| \leq M|u|$, uniformly in $x$. From this we immediately deduce that each of the terms $|P_n|$, $|I_n|$, $|J_n|$, $|K_n|$, $|L_n|$ is less than or equal to $CM/n$, and (1.5) is proved.

Suppose now that instead of the inequality $|g(x+h)-g(x)| \leq M|h|$ we have

$$ (2.1) \quad M_p[g(x+h) - g(x)] = \left\{ \int_0^{2\pi} |g(x+h) - g(x)|^p dx \right\}^{1/p} \leq M|h| $$

for some $p \geq 1$. Then Minkowski's inequality for integrals shows that $M_p[P_n]$, $M_p[I_n]$, $M_p[J_n]$, $M_p[K_n]$, $M_p[L_n]$ are all less than or equal to $CM/n$. For example,

$$ M_p[P_n] \leq \int_0^{\pi/n} M_p[\psi(x, t)] t^{-1} dt \leq \int_0^{\pi/n} 2M dt = 2M\pi/n, $$

$$ M_p[I_n] \leq Cn^{-2} \int_{\pi/n}^{\pi} M_p[\psi(x, t)] t^{-3} dt \leq 2CMn^{-2} \int_{\pi/n}^{\pi} t^{-2} dt = CM/n, $$

and similarly in other cases. Thus, under the hypothesis (2.1),

$$ M_p[\sigma_{n-1}(x) - \xi(x)] \leq BM/n $$

where $B$ is an absolute constant. By an argument similar to that by which Theorem 1 was deduced from the lemma, we obtain the following theorem.

**Theorem 2.** Suppose that the Fourier series of $f(x)$ is of the power series type. Then

$$ M_p[\sigma_{n-1}(x) - f(x)] \leq A\omega_p(2\pi/n) \quad (p \geq 1) $$

where $\omega_p(\delta) = \sup_{|t| \leq \delta} M_p[f(x+t) - f(x)]$.

Theorems 1 and 2 hold for $(C, \alpha)$ means, whatever $\alpha > 0$. The analogues for Abel means are immediate consequences of the Cauchy-Riemann equations.

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