NOTE ON INTERPOLATION FOR A FUNCTION OF
SEVERAL VARIABLES

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The simplest interpolation formula for a function of \( \omega \) variables \( x, y, \cdot \cdot \cdot, z \) is the multiple Gregory-Newton formula, which approximates the function by a polynomial in \( p, q, \cdot \cdot \cdot, r \) of total degree \( n \), namely,

\[
f(x + ph_1, y + qh_2, \cdot \cdot \cdot, z + rh_\omega)
\]

\[
= \sum_{i+j+\cdot \cdot \cdot+b=0}^{n} \left( \begin{array}{l} p \\ i \end{array} \right) \left( \begin{array}{l} q \\ j \end{array} \right) \cdots \left( \begin{array}{l} r \\ k \end{array} \right) \Delta^{i+j+\cdot \cdot \cdot+b}_{x^i y^j \cdot \cdot \cdot z^k} f(x, y, \cdot \cdot \cdot, z),
\]

where \( x, y, \cdot \cdot \cdot, z \) denote the independent variables, \( h_m \) denotes the tabular intervals,

\[
\left( \begin{array}{l} p \\ i \end{array} \right) \] denotes \( \frac{p!\cdots(p-i+1)}{i!} \), with \( \left( \begin{array}{l} p \\ 0 \end{array} \right) = 1, \]

and \( \Delta^{i+j+\cdot \cdot \cdot+b}_{x^i y^j \cdot \cdot \cdot z^k} f(x, y, \cdot \cdot \cdot, z) \) denotes the mixed partial advancing difference of \( f(x, y, \cdot \cdot \cdot, z) \), of order \( i \) with respect to \( x \), \( j \) with respect to \( y \), and so on. The summation is for all sets of values of \( i, j, \cdot \cdot \cdot, k \) such that \( i+j+\cdot \cdot \cdot+k \) goes from 0 to \( n \). Using the notation \( f_s, t, \cdot \cdot \cdot, u \) to denote \( f(x+sh_1, y+th_2, \cdot \cdot \cdot, z+uh_\omega) \), it is apparent that the multiple Gregory-Newton formula involves all values \( f_s, t, \cdot \cdot \cdot, u \) such that \( s+t+\cdot \cdot \cdot+u=0, 1, 2, \cdot \cdot \cdot, n \). Thus for the case of 2 dimensions the arguments are the \( (n+1)(n+2)/2 \) points forming a right triangle, vertex at \( (x, y) \), and for 3 dimensions the arguments are the \( (n+1)(n+2)(n+3)/6 \) points forming a solid tetrahedron, vertex at \( (x, y, z) \).

The purpose of the present note is to show that when (1) is expressed in the simpler form

\[
(2) \quad f(x + ph_1, y + qh_2, \cdot \cdot \cdot, z + rh_\omega) = \sum_{s+t+\cdot \cdot \cdot+u=0}^{n} C_s, t, \cdot \cdot \cdot, u f_s, t, \cdot \cdot \cdot, u,
\]

then we have

\[
(3) \quad C_s, t, \cdot \cdot \cdot, u = \binom{n-p-q-\cdot \cdot \cdot-r}{n-s-t-\cdot \cdot \cdot-u} \binom{p}{s} \binom{q}{t} \cdots \binom{r}{u}.
\]

Thus (1) can be employed without the labor of finding all the mixed

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279
partial differences, which represents a very convenient simplification in the use of the multiple Gregory-Newton formula.

To prove (3) consider the function

\[
\binom{n - x - y - \cdots - z}{n - s_1 - t_1 - \cdots - u_1} \binom{x}{s_1} \binom{y}{t_1} \cdots \binom{z}{u_1},
\]

where \(s_1, t_1, \cdots, u_1\) are any set of non-negative integers whose sum is not greater than \(n\). This function is a polynomial in \(x, y, \cdots, z\) of total degree \(n\) and (2) holds exactly. Applying (2) for \(x = y = \cdots = z = 0, h_1 = h_2 = \cdots = h_u = 1\), it is apparent that with the exception of \(f_{s_1, t_1, \cdots, u_1} = 1\), all the other quantities \(f_{s, t, \cdots, u}\) vanish, because if some \(s, t, \cdots, u\) is less than a respective \(s_1, t_1, \cdots, u_1\), or if every \(s, t, \cdots, u\) is greater than or equal to a respective \(s_1, t_1, \cdots, u_1\) with at least one greater than, then \(f_{s, t, \cdots, u}\) will have a factor

\[
\binom{a}{b}, \quad a \text{ and } b \text{ integers},
\]

\(b > a\), which is 0. This establishes (3).

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\[1\] This line of proof was suggested by Professor W. E. Milne. Another longer proof is by induction, making use of the properties of \(\binom{n}{k}\) and Newton's backward-difference interpolation formula.