The topic of this paper is the extension of the basic facts of valuation theory to noncommutative systems. The purpose of this generalization is twofold. First, the theory of valuations with commutative groups of values is placed in the framework of the theory of \( \Gamma \)-groups, and secondly the general theory leads to the construction of a new class of infinite division algebras. These division algebras are of highly transcendental structure over their respective centers; moreover they may be considered, in special cases, as crossed transcendental extensions of other division algebras.

It is necessary to recall some facts on \( \Gamma \)-groups. A group \( \Gamma \) is called a simply ordered \( \Gamma \)-group if the following axioms are satisfied:

(I) There is defined a binary inclusion relation which is "homogeneous" in the sense that \( \alpha \geq \beta \) implies \( \rho + \alpha + \sigma \geq \beta + \sigma + \sigma \) for all \( \rho, \sigma \),

(II) \( \Gamma \) is a lattice with respect to the ordering relation, and

(III) given \( \alpha, \beta \), either \( \alpha \geq \beta \) or \( \beta \geq \alpha \).

Furthermore \( \alpha \geq \beta \) means \( \alpha \cup \beta = \alpha \). The totality of all positive elements of \( \Gamma \) is a semi-group and shall be denoted by \( \Gamma^+ \). The absolute value \( |\alpha| \) of \( \alpha \) is defined as \( \alpha \cup -\alpha \). Hence \( |\alpha| \) is equal to \( \alpha \) or \(-\alpha\) according as \( \alpha \) lies in \( \Gamma^+ \) or the complement \( \Gamma - \Gamma^+ \).

DEFINITION. A one-valued function \( V \) on a division ring \( D \) upon an \( \Gamma \)-group \( \Gamma \) is called a valuation if the following postulates hold:

\begin{itemize}
  \item Birkhoff, loc. cit. pp. 299, 300, 312.
  \item Birkhoff, loc. cit. pp. 302, 308, 309.
  \item Birkhoff, loc. cit. pp. 309–311.
\end{itemize}

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Each \( \alpha \in \Gamma \) has the form \( V(a) \) for at least one element \( a \in D \).

(ii) \( V(0) \geq \alpha \) for all \( \alpha \in \Gamma^+ \).

(iii) \( V(ab) = V(a) + V(b) \).

(iv) \( V(a+b) \geq \min \{ V(a), V(b) \} \).

**Remark 1.** If \( \Gamma \) is noncommutative then \( D \) is certainly not a field. This follows, by contradiction, from (i) and (iii).

**Remark 2.** All elements \( u \) of \( D \) with \( V(u) = 0 \) form a normal subgroup \( U^* \) of \( D \) for which \( D^*/U \cong \Gamma \). This holds since \( V \) is a homomorphism of \( D^* \) upon \( \Gamma \).

As in the ordinary valuation theory it is now shown, using (iii) and (iv), that the totality \( \mathcal{O} \) of all \( a \in D \) with \( V(a) \geq 0 \) is a ring, the valuation ring of \( V \).

**Lemma 1.** For \( a, b \) in \( \mathcal{O} \), the following statements are equivalent:

(i) \( a = c_1b \) with \( c_1 \) in \( \mathcal{O} \).

(ii) \( a = bc_2 \) with \( c_2 \) in \( \mathcal{O} \).

(iii) \( V(a) \geq V(b) \).

**Proof.** Application of \( V \) to (i) or (ii) yields (iii). Conversely (iii) implies the existence of elements \( \gamma_1 \) and \( \gamma_2 \) in \( \Gamma^+ \) such that \( V(a) = \gamma_1 + V(b) \) and \( V(a) = V(b) + \gamma_2 \). Then there exist elements \( d_1 \) and \( d_2 \) with \( V(d_1) = \gamma_1 \) and \( V(d_2) = \gamma_2 \). Hence \( a = u_1d_1b = bd_2u_2 \) where \( u_1d_1 \) and \( u_2d_2 \) lie in \( \mathcal{O} \), \( u_1 \) and \( u_2 \) in \( U \).

**Lemma 2.** Each ideal \( \mathfrak{I} \) of \( \mathcal{O} \) is two-sided.

**Proof.** Suppose that \( \mathfrak{I} \) is a left ideal, that is, \( \mathcal{O}\mathfrak{I} \subseteq \mathfrak{I} \). Let \( V\mathfrak{I} \) be the set of all \( V(a) \) with \( a \in \mathfrak{I} \). This set is an upper class of \( \Gamma^+ \) as follows by Lemma 1. Now let \( d = \sum_{j=1}^{N} a_j \beta_j \) be an arbitrary element of the set \( \mathfrak{I}\mathcal{O} \); \( a_j \in \mathfrak{I} \), \( \beta_j \in \mathcal{O} \). Then \( a_j \beta_j = b_j^1 a_j, b_j^1 \in \mathcal{O} \) by Lemma 1. Consequently \( d \in \mathfrak{I} \). Thus \( \mathfrak{I} \) is a right ideal.

Now let \( \mathfrak{P} \) be the ideal of elements \( b \in \mathcal{O} \) with \( Vb > 0 \). Moreover, \( \mathcal{O}/\mathfrak{P} \) is a division ring \( D \) for the elements of \( D - 0 = D^* \) can be represented by elements in the multiplicative group \( U \). As usual \( \mathfrak{P} \) is termed the prime ideal of \( V \).

**Lemma 3.** The sets \( \mathcal{O} \) and \( \mathfrak{P} \) are invariants for the group of inner automorphisms of \( D \).

**Proof.** Let \( d \in D^* \) and \( a \in \mathcal{O} \). Then \( V(d^{-1}ad) = -V(d) + V(a) + V(d) \geq 0 \) by the invariance of \( \Gamma^+ \). Hence \( d^{-1}ad \in \mathcal{O} \). Next one finds \( V(d^{-1}ad) > 0 \) for \( a \in \mathfrak{P} \). The possibility \( V(d^{-1}ad) = 0 \) is excluded for \( -V(d) + V(a) + V(d) = 0 \) would imply \( V(a) + V(d) = V(d) \), \( V(a) = 0 \), in contradiction to the assumption on \( a \).
**Lemma 4.** Let \(a, b \in D^*\), then at least one of the pairs \(\{ab^{-1}, b^{-1}a\}\), \(\{a^{-1}b, ba^{-1}\}\) lies in \(\mathfrak{O}\).

**Proof.** Suppose that \(ab^{-1} \in \mathfrak{O}\). Then also \(b^{-1}a \in \mathfrak{O}\) by Lemma 1. Now assume \(ab^{-1} \notin \mathfrak{O}\). Then, by the definition of \(\mathfrak{O}\), \(0 > V(ab^{-1})\). Consequently \(-V(a) + V(b) > 0\) and thus \(a^{-1}b\) and \(ba^{-1}\) lie in \(\mathfrak{O}\) by Lemma 1. Moreover, \(b^{-1}a \notin \mathfrak{O}\) for otherwise \(-V(b) + V(a) \geq 0\) or \(V(a) - V(b) \geq 0\) contrary to the assumption on the elements \(a\) and \(b\).

**Remark.** If \(d \in D\) then either \(d \in \mathfrak{O}\) or \(d = a^{-1}\) where \(a \in \mathfrak{O}\). For the latter observe that \(0 > V(d)\) implies \(-V(d) > V(a^{-1}) + V(d) = V(1) = 0\).

**Lemma 5.** If \(\mathfrak{O}\) is an invariant subring of \(D\) such that for any \(a \in D\), either \(a\) or \(a^{-1}\) is in \(\mathfrak{O}\), then \(\mathfrak{O}\) is a valuation ring for some valuation of \(D\).

**Proof.** The invariance of the ring \(\mathfrak{O}\) implies the invariance of its group of units \(U\). For if \(u \in U\), then \(d^{-1}ud = (d^{-1}ud)^{-1}\) both lie in \(\mathfrak{O}\) for every \(d \in D\), that is, \(d^{-1}ud \in U\). Now let \(\mathfrak{P}\) be the complement \(\mathfrak{O} - U\). The set \(\mathfrak{P}\) is invariant under the group of inner transformations of \(D\). To define the valuation \(V\) for which \(\mathfrak{O}\) is the valuation ring set \(V(a) = aU\) for \(a \in D^*\) and \(V(u) = 0\) for \(u \in U\). The factor group \(D^*/U\) may then be considered as an additive group \(\Gamma\). The group turns out to be a simply ordered \(l\)-group, if \(\alpha = V(a) > \beta = V(b)\) in case \(ab^{-1}\) and \(b^{-1}a\) lie in \(\mathfrak{P}\). As in the commutative theory it now follows that \(\mathfrak{O}\) is the valuation ring for \(V\).

The preceding properties of the valuation \(V\) lead to another description of a valuation. By the homomorphism \(a \mapsto a \mod \mathfrak{P} = H(a) = a \in D\) exactly the valuation ring \(\mathfrak{O}\) is mapped upon \(D\). It is customary to agree that \(H(d) = \infty\) if \(d \notin \mathfrak{O}\). This can happen only if \(H(d^{-1}) = 0\), for \(d \in \mathfrak{O}\) means \(d = a^{-1}\) with \(V(a) > 0\), that is, \(a \notin \mathfrak{P}\), and thus \(H(a) = 0\). Finally \(H(d^{-1}ad) \neq \infty\) if and only if \(H(a) \neq \infty\) for inner automorphisms preserve non-negativeness.

**Theorem 1.** Let \(H\) be a homomorphism of a division ring \(D\) upon a division ring \(D\) and a symbol \(\infty\) so that (i) \(H(a + b) = H(a) + H(b)\) and \(H(ab) = H(a)H(b)\) for any pair \(a, b \in D\) with \(H(a) \neq \infty, H(b) \neq \infty\), (ii) \(H(a) = \infty\) if and only if \(H(a^{-1}) = 0\), and (iii) \(H(d^{-1}ad) \neq \infty\) for all \(d \in D^*\) if and only if \(H(a) \neq \infty\). Then \(H\) arises from a valuation \(V\) of \(D\).

**Proof.** Denote by \(H^{-1}(\mathfrak{S})\) the inverse image of a subset \(S \subseteq D\). Then \(H^{-1}(D) = \mathfrak{O}\) is a ring with \(H^{-1}(D - 0) = U\) for its subgroup of units. Certainly \(\mathfrak{O}\) and \(U\) are invariant under the group of inner

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6 Observe, for example, that \(V(a) \geq 0\) if and only if \(a \in \mathfrak{O}\), \(d^{-1}ad \in \mathfrak{O}\) for \(d \in D^*\) implies that \(\Gamma\) is homogeneous.

7 Observe that for any \(a, b\) in \(D^*\) either \(a^{-1}b\) or \(b^{-1}a\) lies in \(\mathfrak{O}\). Say the former holds, then \(a(a^{-1}b)a^{-1} = ba^{-1}\) also lies in \(\mathfrak{O}\).
transformations. For the first assertion observe that \( a \in \mathcal{O} \) means \( H(a) \neq \infty \). Thus \( H(d^{-1}ad) \neq \infty \) for each \( d \in D^* \) by assumption (iii), consequently \( d^{-1}ad \in \mathcal{O} \). In the second case observe \( H(u) \neq 0 \) for \( u \in U \). If \( d^{-1}ud \notin U \) then \( H(d^{-1}ud) = 0 \), consequently \( H(d^{-1}ud^{-1}d) = \infty \), by assumption (ii). Consequently \( H(u^{-1}) = \infty \), by (iii), that is, \( u \notin \mathcal{O} \) contrary to the assumption on \( u \). Now let \( d \in D^* \), then either \( H(d) \neq \infty \) or \( H(d) = \infty \). By construction of \( \mathcal{O} \) we have \( d \in \mathcal{O} \) if \( H(d) \neq \infty \) and, by (ii), \( d^{-1}a \in \mathcal{O} \) if \( H(d) = \infty \). Thus each element of \( D^* \) is a quotient of elements in \( \mathcal{O} \). Finally let \( a, b \in D^* \), then at least one of the pairs \( \{ ab^{-1}, b^{-1}a \} \), \( \{ a^{-1}b, ba^{-1} \} \) lies in \( \mathcal{O} \). Without loss of generality we may assume \( H(ab^{-1}) \neq \infty \). Then \( H(b^{-1}ab^{-1}) = H(b^{-1}a) \neq \infty \) by assumption (iii). Therefore \( \mathcal{O} \) is a valuation ring by Lemma 5.

Let \( \Delta \) be an isolated subgroup of \( \Gamma \). Then a proper restatement of the commutative proof may be used to show the following lemma.

**Lemma 6.** The prime ideals \( \mathfrak{p} \) of the valuation ring \( \mathcal{O} \) are in 1-1 correspondence with the isolated subgroups \( \Delta \) of \( \Gamma \).

**Corollary.** The invariant isolated subgroups of \( \Gamma \) are in 1-1 correspondence with the prime ideals of \( \mathcal{O} \) which are invariant under all inner automorphisms of \( D \).

**Lemma 7.** Each invariant prime ideal \( \mathfrak{p} \) of \( \mathcal{O} \) determines a quotient ring \( \mathcal{O}_\mathfrak{p} \) in which the extended ideal \( \mathfrak{p}\mathcal{O}_\mathfrak{p} \) is a prime ideal whose residue class ring \( \mathcal{O}_\mathfrak{p}/\mathfrak{p}\mathcal{O}_\mathfrak{p} \) is a division ring.

**Proof.** Let \( \mathcal{O}_\mathfrak{p} \) be the set \( \{ ab^{-1}; a \in \mathcal{O}, b \in \mathcal{O} - \mathfrak{p} \} \). Then \( \mathcal{O}_\mathfrak{p} \) may alternately be defined as the set of quotients \( b^{-1}a \), say \( \mathcal{O}' \). Observe \( b^{-1}a = a^{-1}(ab^{-1})a = a(a^{-1}ba)^{-1} \) for \( a \neq \mathfrak{p} \) with \( a^{-1}ba \notin \mathcal{O} - \mathfrak{p} \) for \( \mathfrak{p} \) is an invariant prime ideal. Thus \( \mathcal{O}' \subseteq \mathcal{O}_\mathfrak{p} \) and conversely. Next \( \mathcal{O}_\mathfrak{p} \) is an invariant subring of \( D \), for \( d^{-1}(ab^{-1})d = (d^{-1}ad)(d^{-1}bd)^{-1} \) where \( d^{-1}ad, d^{-1}bd \in \mathcal{O} - \mathfrak{p} \) with \( d \in D^* \) since \( \mathcal{O} \) and \( \mathfrak{p} \) are invariant. It remains to show that \( \mathcal{O}_\mathfrak{p} \) is a ring. Let \( a_1b_1^{-1}, a_2b_2^{-1} \) be two elements of \( \mathcal{O}_\mathfrak{p} \). Without loss of generality it may be assumed that \( V(b_2) \leq V(b_1) \), then \( a_1b_1^{-1}a_2b_2^{-1} = (a_1 + a_2b_2^{-1}b_1)b_1^{-1} \) where \( b_2^{-1}b_1 \in \mathcal{O} \). Consequently the sum lies in \( \mathcal{O}_\mathfrak{p} \). For the product observe \( a_1b_1^{-1}a_2b_2^{-1} = a_1b_1^{-1}b_2^{-1}a_2 \) with \( b_2 \in \mathcal{O} - \mathfrak{p} \) by the invariance of \( \mathcal{O} \) and \( \mathfrak{p} \). Next \( a_1(b_2^{-1}b_1)^{-1}a_2 = b_2^{-1}a_1a_2 \in \mathcal{O}_\mathfrak{p} \) with \( b_2 \in \mathcal{O} - \mathfrak{p} \), \( a_2 \in \mathcal{O} \), by the invariantive properties.

Consider next the extended set \( \mathcal{O}_\mathfrak{p} = \{ \sum \lambda_i \mathfrak{p} \alpha_i \beta_i^{-1} \} \) with \( \mathfrak{p} \in \mathfrak{p}, \alpha, \beta \in \mathcal{O}_\mathfrak{p} \). Then \( \mathfrak{p} \alpha \beta^{-1} = \alpha \beta j \) with \( x_j = p \alpha p \beta^{-1} p^{-1} = (p \alpha \beta^{-1})(p^{-1} b \beta p) \) with the first factor in \( \mathcal{O} \) and the second in \( \mathcal{O} - \mathfrak{p} \). Thus \( \mathfrak{p} \mathcal{O}_\mathfrak{p} = \mathcal{O}_\mathfrak{p} \mathfrak{p} \). The general element \( \sum \mathfrak{p} \alpha \beta^{-1} \) may be written as \( \mathfrak{p} \sum (p \beta^{-1} \alpha \beta) = \mathfrak{p} s_1, s_1 \in \mathcal{O}_\mathfrak{p} \), for it may be assumed without loss of generality that
$V(p_1) \leq V(p_j)$, $j = 2, \ldots, N$. As an immediate consequence it follows that $p\mathcal{O}_p$ is an invariant set for $d^{-1}(p_is_i)d = (d^{-1}p_id)(d^{-1}s_id) \in p\mathcal{O}_p$ since $p$ and $\mathcal{O}_p$ were recognized to be invariant sets. Let $p_is_i, p_is_2 \in p\mathcal{O}_p$. Without loss of generality it may be assumed that $V(p_1) \leq V(p_2)$. Then $p_is_1 + p_is_2 = p_is_1 + p_is_2 \in p\mathcal{O}_p$, therefore $s_1 + s_2 \in \mathcal{O}_p$ and $p_is_1 + p_is_2 \in p\mathcal{O}_p$. Finally $p\mathcal{O}_p \cdot \mathcal{O}_p \subseteq p\mathcal{O}_p$ and $\mathcal{O}_p \cdot p\mathcal{O}_p \subseteq p\mathcal{O}_p$, that is, $p\mathcal{O}_p$ is a two-sided ideal. To show that $p\mathcal{O}_p$ is a prime ideal it suffices to prove that $a_1b_1^{-1}, a_2b_2^{-1} \in \mathcal{O}_p - p\mathcal{O}_p$ implies $a_1b_1^{-1}a_2b_2^{-1} \in \mathcal{O}_p - p\mathcal{O}_p$. Observe $a_1, a_2 \in \mathcal{O} - p$, that is, $a_1a_2 \in \mathcal{O} - p$. Therefore $a_1b_1^{-1}a_2b_2^{-1} = a_1a_2(b_2^{-1}b_1a_2^{-1})$ where $b_1, a_2b_2^{-1} \in \mathcal{O} - p$ for $p$ is an invariant prime ideal of $\mathcal{O}$. Consequently $a_1b_1^{-1}a_2b_2^{-1} \in \mathcal{O}_p - p\mathcal{O}_p$. Let $\mathcal{O}_p/p\mathcal{O}_p = D_0$, and suppose $c_0 \in D_0$. Then there exist $a, b$ so that $(ab^{-1})_0 = c_0$ with $a \in \mathcal{E}_p$. Hence $(ab^{-1})^{-1}ba^{-1} \in p\mathcal{O}_p$, thus $(ba^{-1})_0 \in D_0$ and $c_0(ba^{-1})_0 = (1)_0$.

Now it is possible to carry over the results of the commutative case.

**Lemma 8.** Each invariant prime ideal $p$ of $\mathcal{O}$ with the associated isolated subgroup $\Delta$ of $\Gamma$ determines a valuation $V_\Delta$ in $D_\Delta = \mathcal{O}_p/p\mathcal{O}_p$ whose valuation ring is $\mathcal{O}_p/p$ and whose value group is $\Delta$. More specifically, $V_\Delta(a_0) = V(a) \mod \mathcal{O}_p = a_0 \in D_\Delta$.

**Lemma 9.** The quotient ring $\mathcal{O}_p$ is a valuation ring of $D$ whose value group is $\Gamma/\Delta$ letting $V_{\Gamma/\Delta}(a) = V(a) \mod \Delta$, and whose associated residue algebra is $D_\Delta$.

The preceding lemmas may be combined so as to give the following theorem.

**Theorem 2.** Relative to each isolated invariant subgroup $\Delta$ of $\Gamma$ the valuation $V$ can be split into (i) a valuation $V_{\Gamma/\Delta}$ of $D$ and (ii) a valuation $V_\Delta$ of the residue algebra $D_\Delta$ for $V_{\Gamma/\Delta}$ with its value group equal to $\Delta$.

**Remark.** The prime ideals $\delta$, of $D_\Delta$ in $\mathcal{O}_p/\mathcal{O}_p$ arise exactly as the homomorphic images of the prime ideals $p, \subseteq \mathcal{O}$ with $p, \subseteq \mathcal{O}_p$. Since every invariant subgroup of $\Delta$ is not an invariant subgroup of $\Gamma$ it will in general not be true that an invariant prime $\delta$, is the homomorphic image of an invariant prime ideal $p$, of $\mathcal{O}$.

**Theorem 3.** Suppose $D$ has a valuation $V_1$ with value group $\Gamma_1$ and residue class algebra $D_1$. Then each valuation $V_2$ in $D_1$ with value group $\Gamma_2$ whose valuation ring has an invariant image in $D$ determines a valuation $V$ on $D$ with value group $\Gamma$ so that $\Gamma$ contains an order isomorphic image of $\Gamma_2$ with $\Gamma/\Gamma_2 \cong \Gamma_1$. Moreover the residue class ring of $D$ with respect to $V$ is equal to the residue class ring of $D_1$ with respect to $V_2$. 
The proof of the theorem involves a direct restatement of the proof for the parallel theorem involving a commutative field.

The preceding discussion of a division algebra with a valuation $V$ may be utilized to establish the existence of a wide variety of algebras with prescribed value groups and algebras of residue classes. Suppose that $D$ is a division algebra which is to be the algebra of residue classes for a valuation $V$ with value group $\Gamma$. Noting that in a given algebra $D$ the elements $d \in D^*$ induce by $a \mod \mathfrak{m} \rightarrow d^{-1}ad \mod \mathfrak{m}$, $a \in \mathfrak{m}$, automorphisms on the algebra of residues $\overline{D}$, one is led to the following construction. Assume that each $\gamma \in \Gamma^+$ induces an automorphism $\bar{a} \rightarrow \bar{a}^\gamma$ in $\overline{D}$ and let to $\gamma$ be associated a symbol $t(\gamma)$. Let $D$ be isomorphic to $\overline{D}$. Consider then a group extension of $D^*$ by the group $\Gamma$ with the defining relations

$$a \rightarrow a^\gamma = t(\gamma)^{-1}at(\gamma), \quad t(0) = 1, \quad t(\gamma)^{-1} = t(-\gamma),$$

$$t(\alpha)t(\beta) = f(\alpha, \beta)t(\alpha + \beta),$$

where the $f(\alpha, \beta)$ satisfy the customary relations for factor sets. Now define $D$ to be the set of all formal power series $A = \sum a_i t(\alpha_i)$ where $a_i \in D$ and $\{\alpha_i\}$ is a well ordered monotonically increasing sequence in $\Gamma$ with a finite first element. If $B = \sum b_i t(\beta_i)$ is another element of $D$ then $A + B$ is to be the series obtained by adding the coefficients at identical marks. The product $AB$ is to be defined by formal multiplication observing that $at(\alpha)bt(\beta) = ab^*t(\alpha, \beta)t(\alpha + \beta)$. Thus the system $D$ becomes a ring without divisors of zero. Next define $V(A) = \alpha_1 = V(t(\alpha_1))$ where $A = \sum a_i t(\alpha_i) = [\sum a_i t(\alpha_i)t(\alpha_i^{-1})]t(\alpha_1) = [\sum a_i f(\alpha_i, -\alpha_1)t(\alpha_i - \alpha_1)]t(\alpha_1), \alpha_i - \alpha_1 \geq 0$. Then $V(\alpha) = 0$ for $\alpha \in D^*$. The function $V$ satisfies the postulates for a valuation. Observe that $A = at(\alpha) + A_0, B = bt(\beta) + B_0$ with $V(A_0) > \alpha, V(B_0) > \beta$, respectively, imply $AB = ab^*t(\alpha, \beta)t(\alpha + \beta) + C_0$ where $V(C_0) > \alpha + \beta$. Hence $V(AB) = \alpha + \beta$. Finally $V(A + B) \geq \min \{V(A), V(B)\}$. For the proof one may assume without loss of generality that $\beta \geq \alpha$. Then $A + B = [a + A\theta(\alpha)^{-1} + b\theta(\beta, -\alpha)t(\beta - \alpha) + B\theta(\alpha)^{-1}]t(\alpha)$ which proves the triangle inequality. It remains to show that each element $A \in D^*$ has an inverse. Observe that each sequence $\{\sum_{j=0}^n C_i^j, n \rightarrow \infty, V(C) > 0\}$ has a limit $\sum_{j=0}^\infty C_i^j$ in the set $D$. Thus $(1+C)^{-1} = \sum_{j=0}^\infty (-1)^j C_i^j$ lies in $D$. Write $A = a(1+C)t(\alpha), a \in D^*$, then $A^{-1} = t(-\alpha)\sum_{j=0}^\infty (-1)^j C_i^j a^{-1}$. It is shown directly that the valuation ring $\mathfrak{O} = \{\sum a_i t(\alpha_i), \alpha_i \geq 0\}$
contains the prime ideal \( \mathfrak{p} = \{ \sum a_i \alpha_i \mid \alpha_i > 0 \} \) so that \( \mathcal{O}/\mathfrak{p} \cong D/\mathfrak{p} \). Moreover \( \mathcal{O} \) and \( \mathfrak{p} \) are invariant subsets of \( D \).

**Definition 2.** A division algebra \( D \) is termed relatively complete with respect to a valuation \( V \) if each of its subfields \( K \) containing the center \( Z \) is relatively complete with respect to the induced valuation \( V_K \).

**Lemma 10.** A relatively complete division algebra of finite rank \( n^2 \) over its center \( Z \) has a commutative value group with respect to the given valuation.

**Proof.** Observe first that the valuation \( V \) induces a non-trivial valuation \( V_Z \) on \( Z \). For let \( d^m + z_1 d^{m-1} + \cdots + z_m = 0, z_j \in Z \), be the irreducible equation satisfied by an element \( d \in D \) with \( V(d) > 0 \). Then \( V(z_m) = V(d) + V(d^{m-1} + z_1 d^{m-2} + \cdots + z_{m-1}) \). Thus \( V(z_m) = V_Z(z_m) > 0 \) in case \( V_Z(z_j) \geq 0, j = 1, \ldots, m-1 \), by the triangle inequality for valuations. In case some \( V_Z(z_j) < 0 \) nothing is to be proved. Since \( Z \) is, by hypothesis, relatively complete with respect to \( V_Z \) the usual theory of prolongation for valuations may be applied.\(^9\) Set \( V^*(a) = n^{-1} V_Z(Na) \) where \( N \) denotes the reduced norm of \( D/Z \). Then \( V^*(a) = V(a) \) for otherwise the subfields \( Z(a) \), \( Z \) would be relatively complete with respect to two inequivalent valuations. Hence \( Z \) would be algebraically complete contrary to the hypothesis that \( D \) is a division algebra.\(^10\) Next let \( a, b \in D \), then \( V(ab) = V^*(ab) = V^*(ba) = V(ba) \), that is, the value group \( \Gamma \) of \( V \) is abelian.

**Definition 3.** A relatively complete division algebra \( D \) is termed algebraic if the algebra \( D(a, b) \) generated by any two elements \( a, b \in D \) over \( Z \) has finite rank over its center \( Z(a, b) \).

**Theorem 4.** The value group of an algebraic relatively complete division algebra \( D \) is abelian.

**Proof.** Let \( \alpha, \beta \) be two elements of \( \Gamma \). Suppose that \( a, b \) are any two elements of \( D \) with \( V(a) = \alpha, V(b) = \beta \). By hypothesis the algebra \( D(a, b) \) has finite rank over the relatively complete field \( Z(a, b) \). Hence, by Lemma 10, \( V(ab) = V(ba) \), that is, \( \alpha + \beta = \beta + \alpha \).

**Remark.** The preceding theorem indicates that a division algebra with a noncommutative value group must contain elements which are transcendental over its center.

**Illustrative Example.** Let \( \Gamma \) be the lexicographically ordered group of motions in the plane, that is, the group of all couples of real

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\(^10\) See Albert, loc. cit.
numbers \((\alpha, \beta)\) subject to the following law of combination: if \((\gamma, \delta)\) is a second pair then \((\alpha, \beta) + (\gamma, \delta) = (\alpha + \gamma, e^\gamma \beta + \delta)\). The set \(\Gamma^+\) of positive elements consists of all couples for which either \(\alpha > 0\) or \(\alpha = 0\) and \(\beta > 0\). Observe the following facts: (i) \((-\alpha, \beta) = (-\alpha, -\beta e^{-\alpha})\), (ii) the set \(\{(0, \beta)\}\) is an invariant isolated subgroup \(\Delta\), (iii) \(\Gamma/\Delta \cong \{\alpha\}\), and (iv) \(\Gamma\) has no proper center. Let \(F\) be a field and \(t\) a transcendental element over \(F\). Consider the set \(D\) of all formal power series \(\sum (\alpha, \beta) a_{\alpha, \beta} t^{(\alpha, \beta)}\) where \(a_{\alpha, \beta} \in F\) and the elements \((\alpha, \beta)\) form well ordered monotonically increasing sequences. Set \(t^{(0,0)} = 1\) and \(t^{(\alpha, \beta)} t^{(\gamma, \delta)} = t^{\xi}\), \(\xi = (\alpha, \beta) + (\gamma, \delta)\). Then \(D\) is a division algebra with \(\Gamma\) as a value group. The valuation ring \(O\) consists of all series \(\sum a_{\alpha, \beta} t^{(\alpha, \beta)}\) where all \((\alpha, \beta) \geq (0, 0)\); the prime ideal \(\mathfrak{P}\) contains all series with \((\alpha, \beta) > (0, 0)\). Both \(O\) and \(\mathfrak{P}\) are invariant subsets for the group of inner automorphisms of \(D^*\), and \(O/\mathfrak{P} \cong F\). Corresponding to \(\Delta\) there is an invariant prime ideal \(\mathfrak{p}\) in \(O\) whose associated valuation ring maps homomorphically on the field of all formal series \(\mathcal{F}_\Delta = \{\sum a_{\alpha, \beta} t^{(0,0)}\} = \{\sum a_{\alpha, \beta} t^{\alpha}\}\). Select now in \(D\) the subfield \(F_\Delta = \{\sum a_{\alpha, \beta} t^{(0,0)}\} = \{\sum a_{\alpha, \beta} t^{\alpha}\}\). Next set \(t^{(\alpha,0)} = t_2^\alpha\), then \(t^{(\alpha,\beta)} = t_\mu^\mu\). The rule of combination in \(\Gamma\) implies \(t_2^\alpha t_2^\beta = t_\mu^\mu\) with \(\mu = e^\alpha \beta\). Since each element of \(D\) may now be expressed as \(\sum_{\alpha < A, A_{1,2}} a_{\alpha, A_{1,2}} t_2^{A_{1,2}}\), \(A_{1,2} \in F_\Delta, \{\alpha, A\}\) increasing, it can be seen that \(D\) is a crossed extension of \(F_\Delta\) by the set \(\{t_2^\alpha\}\). The associated factor set is equal to unity for \(t_2^\alpha t_2^\beta = t_2^\gamma = t_2^{\alpha + \beta}\). This interpretation of the algebra \(D\) as a transcendental crossed extension of the algebra \(F_\Delta\) by means of an extension of the value group of \(F_\Delta\) can be generalized in several directions. As in the construction of an algebra for a given algebra of residue classes \(\overline{D}\) and a value group \(\Gamma\) one may introduce factor sets. Moreover statements can be made for an algebra \(D\) whose value group \(\Gamma\) possesses a chain of normal isolated subgroups \(\Delta_j\) whose factor groups \(\Gamma_j/\Gamma_{j+1}\) are isomorphic to abelian ordered groups. To obtain explicit results it is useful to assume that (i) the successive algebras of residue classes \(D_{j+1}\) have isomorphic images \(\phi_j(D_{j+1})\) in \(D_j\), (ii) the isomorphism \(\phi_j\) commutes with the group of inner automorphisms of \(D_j\), and (iii) \(D_j\) is maximally complete with respect to the valuation having \(\Gamma_j/\Gamma_{j+1}\) for its value group.\(^{11}\)

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\(^{11}\)Observe in this connection the results of Kaplansky, loc. cit.