THE BASIS THEOREM FOR VECTOR SPACES OVER RINGS

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It is the purpose of this note to establish the following theorem:

**Theorem.** A vector space $M = u_1 K + \cdots + u_m K$ of $m$ basis elements over a ring $K = \{0, a, b, \cdots, 1\}$ with unit 1 has the property that every subspace $N > 0$ possesses a basis of $n \leq m$ elements if and only if $K$ is a right principal-ideal-ring without zero-divisors.

That such a ring insures the basis condition for subspaces is well known [3, p. 121].

Suppose now that every subspace $N > 0$ has a basis of $n \leq m$ elements. It has been shown [2, Theorem (F)] that every right ideal $R > 0$ of $K$ must then have a single generator: $R = r_0 K$, where $r_0 k = 0$ implies $k = 0$. Moreover, since every right ideal has a finite set of generators, the ascending chain condition must hold for right ideals of $K$ [3, p. 26]. It therefore suffices to prove the following two lemmas.

**Lemma 1.** In a ring $K$ with unit 1 and ascending chain condition for right ideals, equations $ab = 1$, $ac = 0$ imply $c = 0$.

If $c \neq 0$, the linear transformation $k \mapsto ak$, $k \in K$, would be of type (iv) [2, p. 313], that is, $K/K_0 \cong K$, and $0 < K_0 < K_1 < K_2 < \cdots$, where $K_i$ is defined inductively as the set of all elements of $K$ mapped into elements of $K_{i-1}$. This contradicts the chain condition.

**Lemma 2.** A ring $K$ with unit in which every right ideal $R > 0$ is of the form $r_0 K$, where $r_0 k = 0$ implies $k = 0$, has no zero divisors.

Let $sc = 0$, $s \neq 0$, and $s K = r_0 K \neq 0$, where $r_0 k = 0$ implies $k = 0$. We have $s = r_0 a$, $r_0 = sb = r_0 \cdot ab$, $r_0 (ab - 1) = 0$, and hence $ab = 1$. Also, $sc = 0 = r_0 ac$, and $ac = 0$. Since Lemma 1 applies to $K$, $c = 0$.

It should be noted that the result follows also from a result of Baer's [1, Theorem 5 or Lemma 4] which states that in a ring with unit and weak maximal condition, $ab = 1$ implies $ba = 1$.

**Bibliography**


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1 Numbers in brackets refer to the bibliography.

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ON A CONSTRUCTION FOR DIVISION ALGEBRAS OF ORDER 16

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It is not known whether there exist division algebras of order 16 (or greater) over the real number field $\mathbb{R}$. In discussing the implications of this question in algebra and topology, A. A. Albert told the author that the well known Cayley-Dickson process\(^1\) does not yield a division algebra of order 16 over $\mathbb{R}$ and suggested a modification of that process which might. It is the purpose of this note to show that, while Albert's construction can in no instance yield such an algebra over $\mathbb{R}$, it does yield division algebras of order 16 over other fields, in particular the rational number field $\mathbb{Q}$.

Initially consider an arbitrary field $F$. Let $C$ be a Cayley-Dickson division algebra of order 8 over $F$. Define\(^2\) an algebra of order 16 over $F$ with elements $c=a+vb$, $z=x+vy$ ($a, b, x, y$ in $C$) and with multiplication given by

\[(1) \quad cz = (a + vb)(x + vy) = (ax + g \cdot ybS) + v(aS \cdot y + xb)\]

where $S$ is the involution $x \mapsto xS = t(x) - x$ of $C$ and $g$ is some fixed element of $C$. The Cayley-Dickson process is of course the instance $g = \gamma$ in $F$.

For $A$ to be a division algebra over $F$ the right multiplication\(^1\) $R_z$ must be nonsingular for all $z \neq 0$ in $A$. Now

\[R_z = \begin{pmatrix} R_z & SR_y \\ SL_vL_x & L_z \end{pmatrix}\]

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\(^1\) See [1] and [2] for background and notations. Numbers in brackets refer to the references cited at the end of the paper.

\(^2\) We should remark that this modification of the Cayley-Dickson process does yield non-alternative division algebras of orders 4 and 8 over $\mathbb{R}$ when applied to the algebras of complex numbers and real quaternions instead of to $C$. See R. H. Bruck, Some results in the theory of linear non-associative algebras, Trans. Amer. Math. Soc. vol. 56 (1944) pp. 141–199, Theorem 16C, Corollary 1, for a generalization.