ON A CONSTRUCTION FOR DIVISION ALGEBRAS OF ORDER 16

R. D. SCHAFER

It is not known whether there exist division algebras of order 16 (or greater) over the real number field $\mathbb{R}$. In discussing the implications of this question in algebra and topology, A. A. Albert told the author that the well known Cayley-Dickson process\(^1\) does not yield a division algebra of order 16 over $\mathbb{R}$ and suggested a modification of that process which might. It is the purpose of this note to show that, while Albert's construction can in no instance yield such an algebra over $\mathbb{R}$, it does yield division algebras of order 16 over other fields, in particular the rational number field $\mathbb{Q}$.

Initially consider an arbitrary field $F$. Let $C$ be a Cayley-Dickson division algebra of order 8 over $F$. Define\(^2\) an algebra of order 16 over $F$ with elements $c = a + vb$, $z = x + vy$ ($a$, $b$, $x$, $y$ in $C$) and with multiplication given by

\[
(1) \quad cz = (a + vb)(x + vy) = (ax + g \cdot ybS) + v(aS \cdot y + xb)
\]

where $S$ is the involution $x \mapsto xS = t(x) - x$ of $C$ and $g$ is some fixed element of $C$. The Cayley-Dickson process is of course the instance $g = \gamma$ in $F$.

For $A$ to be a division algebra over $F$ the right multiplication\(^1\) $R_s$ must be nonsingular for all $z \neq 0$ in $A$. Now

\[
R_s = \begin{pmatrix}
R_s & SR_y \\
SL_yL_z & L_z
\end{pmatrix}
\]

Received by the editors January 19, 1945, and, in revised form, March 19, 1945.

\(^1\) See [1] and [2] for background and notations. Numbers in brackets refer to the references cited at the end of the paper.

\(^2\) We should remark that this modification of the Cayley-Dickson process does yield non-alternative division algebras of orders 4 and 8 over $\mathbb{R}$ when applied to the algebras of complex numbers and real quaternions instead of to $C$. See R. H. Bruck, *Some results in the theory of linear non-associative algebras*, Trans. Amer. Math. Soc. vol. 56 (1944) pp. 141–199, Theorem 16C, Corollary 1, for a generalization.
and, if \( g \neq 0 \), \( R_s \) is nonsingular in case either \( x = 0 \), \( y \neq 0 \) or \( x \neq 0 \), \( y = 0 \). Therefore let \( x \neq 0 \), \( y \neq 0 \). Then

\[
\begin{vmatrix}
R_s & SR_y \\
SL_yL_g & L_s
\end{vmatrix} = \begin{vmatrix}
R_s & 0 \\
SL_yL_g & L_s - SL_yL_gR_s^{-1}SR_y
\end{vmatrix}
\]

\[
= \begin{vmatrix}
R_s & L_s - \frac{1}{n(x)} R_ySRS R_gS R_sL_Ry
\end{vmatrix}
\]

\[
= \begin{vmatrix}
R_s & L_s - \frac{1}{n(x)} R_ySRS R_gS R_yL_sR_sL_s
\end{vmatrix}
\]

by a lemma of Moufang.\(^2\) Hence

\[
| R_s | = | L_s | \cdot | n(x)R_yR_y^{-1} | \cdot \left| n(x)R_yR_gR_y - n(g)n(y)R_{xy} \right|. 
\]

That is, \( A \) is a division algebra over \( F \) if and only if the transformation

\( n(x)R_yR_gR_y - n(g)n(y)R_{xy} \)

is nonsingular for all \( x, y \neq 0 \) in \( C \).

Now let \( F = \mathbb{R} \), the field of real numbers. The non-scalar\(^4\) element \( g \) generates \( B \subset Q = B + uB \), \( Q \) a real quaternion algebra, \( u^2 = -n(u) \), \( gu = u \cdot gS \). Multiplication in the Cayley algebra \( C = Q + wQ \) is defined by the right multiplication

\[
R_s + ur = R_{(s, r)} = \begin{pmatrix} R_q & SR_r \\ -SL_r & L_q \end{pmatrix}
\]

for \( q, r \) in \( Q \) and \( S \) the involution \( q \mapsto qS = t(g) - g \) of \( Q \). Specialize two elements \( x, y \) of \( C \) in the following manner. Let \( y = u \in Q \); then \( y^2 = -n(y) \), \( gy = y \cdot gS \). Let \( x \in wQ \) and \( n(x) = \zeta n(y) \) where \( \zeta > 0 \), \( \zeta^2 = n(g) \). Then \( x y = (0, x)R_{(y, 0)} = (0, yx) \) and

\[
| n(x)R_yR_gR_y - n(g)n(y)R_{xy} |
\]

\[
= | n(x) \begin{pmatrix} 0 & SR_y \\ -SL_y & 0 \end{pmatrix} \begin{pmatrix} R_0 & 0 \\ 0 & L_0 \end{pmatrix} \begin{pmatrix} R_y & 0 \\ 0 & L_y \end{pmatrix} - n(g)n(y) \begin{pmatrix} 0 & SR_yz \\ -SL_yz & 0 \end{pmatrix} |
\]

\[
= | n(x)SR_sL_y - n(g)n(y)SR_yz |
\]

\[
= \begin{vmatrix}
R_xL_s & n(x)SR_xL_y - n(g)n(y)R_y \\
0 & n(g)n(y)L_y - n(x)R_yy
\end{vmatrix}
\]

\(^2\) [2, Lemma 1].

\(^4\) The Cayley-Dickson process (the case \( g = \gamma \), a scalar) may be eliminated by this argument too. If \( \gamma \geq 0 \), let \( y = \beta \) in \( \mathbb{R} \), \( n(x) = \gamma \beta^2 \); if \( \gamma < 0 \), let \( y = i, f = j, n(x) = -\gamma \) in what follows.
since $x, y, g$ are quaternions. That is, $|n(x)L_{uy} - n(g)n(y)R_u| = 0$ would imply that transformation (2) is singular.

Choose $f = \xi y + yg$. Then $f\{n(x)L_{uy} - n(g)n(y)R_u\} = n(x)\xi gy + n(x)yg + n(g)n(y)\xi y^2 - n(g)n(y)gy = n(x)y^2gSg - n(g)n(x)y^2 = 0$.

Hence (2) is singular and $A$ is not a division algebra over $\mathbb{R}$.

The easy generalization that there is no choice of $g$ to make $A$ a division algebra of order 16 over any field $F$ should not be made. For the singularity of transformation (2) implies that there exists an element $h \neq 0$ in $\mathbb{C}$ such that $n(x)\{(hx)g\}y = n(g)n(y)h(xy)$. Since the norm of a product is the product of the norms in an alternative division algebra,\(^5\)

$$n(x)^2n(h)n(g)n(y) = n(g)^2n(y)^2n(h)n(x) \quad \text{or} \quad n(x)^2 = n(g)n(y)^2$$

in case $g \neq 0$. That is, the transformation (2) cannot be singular (and $A$ is therefore a division algebra) for any choice of $g$ in $\mathbb{C}$ such that $n(g)$ is not the square of an element in $F$.

For example, let $F$ be in particular the field $\mathbb{R}$ of rational numbers, and $g = 1 + i$ so that $n(g) = 2$. Then the algebra $A$ with multiplication defined by (1) is a division algebra of order 16 over $\mathbb{R}$.

REFERENCES


SAN FRANCISCO, CALIF.

\(^5\) [2, Lemma 2].