

## A PROOF OF A THEOREM ON COMMUTATIVE MATRICES

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The following theorem is well known (see, for example, Wedderburn, *Lectures on matrices*, p. 106):

*“If the matrix  $B$  commutes with every matrix that commutes with  $A$ , then  $B$  is a scalar polynomial of  $A$ .”*

It is thought, however, that the proof given below is simple enough to be of interest. The proof is based on the main theorem for abelian groups with a finite number of generators. The version of this theorem given in van der Waerden, *Moderne Algebra*, vol. 2, pp. 114 and 122, is especially well suited for our purpose. Let  $\mathfrak{M}$  be a finite-dimensional vector space over a commutative field  $K$ . Let  $A$  be a fixed linear endomorphism of  $\mathfrak{M}$ . All endomorphisms of the form  $P(A)$ , where  $P(x)$  is a polynomial with coefficients in  $K$ , form a euclidean ring of operators on  $\mathfrak{M}$ . “Admissible subgroups” (van der Waerden, *Moderne Algebra*, vol. 1, p. 145) with respect to this set of operators are those subspaces of  $\mathfrak{M}$  which are invariant under  $A$ . The main theorem about the decomposition of abelian groups, as applied to  $\mathfrak{M}$ , then reads: There exist a finite number of subspaces  $\mathfrak{M}_i$  and polynomials over  $K$ ,  $P_i(x)$ , such that:

- (1a)  $\mathfrak{M}$  is a direct sum of the  $\mathfrak{M}_i$ .
- (1b)  $\mathfrak{M}_i$  is invariant under  $A$ .
- (1c) Each  $\mathfrak{M}_i$  is cyclic. This means that there exist elements  $e_i$  such that each element of  $\mathfrak{M}_i$  is of the form  $P(A)e_i$ .
- (1d)  $P_i(x)$  generates the annihilating ideal of  $\mathfrak{M}_i$ .
- (1e)  $P_{i+1}(x)$  divides  $P_i(x)$ .

It follows that  $P_1(A) = 0$  and that  $P(A) = 0$  implies that  $P_1(x)$  divides  $P(x)$ . (In a terminology sometimes used  $P_i(x)$  is the order of  $e_i$  with respect to  $A$  and  $P_1(x)$  is the minimal polynomial of  $A$ . Thus the order of  $e_1$  is the minimal polynomial of  $A$ . Conversely, once the existence of an element with this property has been demonstrated, the decomposition theorem is easily proved.)

We denote by  $E_i$  the projection on  $\mathfrak{M}_i$ , that is, the linear endomorphism uniquely defined by:  $E_i f = f$  if  $f$  is in  $\mathfrak{M}_i$  and  $E_i f = 0$  if  $f$  is in  $\mathfrak{M}_j$ ,  $j \neq i$ . It follows that  $f$  is in  $\mathfrak{M}_i$  if and only if  $E_i f = f$ . An endomorphism  $C$  which commutes with  $E_i$  leaves  $\mathfrak{M}_i$  invariant because if  $f$  is in  $\mathfrak{M}_i$ , then  $E_i C f = C E_i f = C f$ . Conversely, if all  $\mathfrak{M}_i$  are invariant un-

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der  $C$ , then  $CE_i = E_i C$ . Also, if  $E$  is the identity mapping, then  $E = \Sigma E_i$ .

After these preliminaries we are ready to give a concise formulation and a proof of the theorem stated at the very beginning of this note:

**THEOREM 1.** *Let  $\mathfrak{M}$  be a finite-dimensional vector space over a commutative field  $K$  and  $A$  and  $B$  linear endomorphisms of  $\mathfrak{M}$  such that  $B$  commutes with every endomorphism that commutes with  $A$ . Then there exists a polynomial  $\bar{Q}(x)$  with coefficients in  $K$  such that  $B = \bar{Q}(A)$ .*

Using the notation explained above we first prove a lemma:

**LEMMA.** *A polynomial  $Q(x)$  may be found satisfying the relation  $Be_1 = Q(A)e_1$  (where as remarked above any element whose order is the minimal polynomial of  $A$  may be taken as  $e_1$ ).  $Q(x)$  also satisfies the relation  $BE_1 = Q(A)E_1$ .*

**PROOF.** Since each  $\mathfrak{M}_i$  is invariant under  $A$ , each  $E_i$  commutes with  $A$  and hence with  $B$ . Thus each  $\mathfrak{M}_i$  is also invariant under  $B$ , in particular  $Be_1$  is in  $\mathfrak{M}_1$  from which the existence of  $Q(x)$  follows by (1c). Now let  $f$  be in  $\mathfrak{M}_1$ . Then for some polynomial  $P(x)$ ,  $f = P(A)e_1$  and  $BE_1 f = Bf = BP(A)e_1 = P(A)Be_1 = P(A)Q(A)e_1 = Q(A)f = Q(A)E_1 f$ . If  $f$  is in  $\mathfrak{M}_i$ ,  $i \neq 1$ , then  $BE_1 f = Q(A)E_1 f = 0$  which concludes the proof of the lemma.

In order to prove that the  $Q(x)$  defined in the lemma may be taken as the  $\bar{Q}(x)$  of Theorem 1, we have to show that  $BE_i = Q(A)E_i$  for any  $i$ . For this purpose we define a mapping  $M_i$  by:

(2a)  $M_i R(A)e_1 = R(A)e_i$  for any polynomial  $R(x)$ .

(2b)  $M_j f = 0$  for  $f$  in  $\mathfrak{M}_j$ ,  $j \neq 1$ .

(2c)  $M$  is a linear endomorphism of  $\mathfrak{M}$ .

Condition 2a defines a one-valued mapping because if  $R(A)e_1 = S(A)e_1$ , then  $R(x) - S(x)$  is divisible by  $P_1(x)$  (by 1d) and by  $P_i(x)$  (by 1e) and hence  $R(A)e_i = S(A)e_i$ . Evidently  $M_i$  commutes with  $A$  and hence with  $B$ . From this follows:  $Be_i = BM_i e_1 = M_i Be_1 = M_i Q(A)e_1 = Q(A)e_i$ . Since  $Be_i = Q(A)e_i$ , we may prove as above that  $BE_i = Q(A)E_i$ .

The proof is now complete since  $B = \Sigma BE_i = \Sigma Q(A)E_i = Q(A)$ .

The above proof suggests a generalization. Namely, in Theorem 1 the vector space  $\mathfrak{M}$  and the ring of polynomials of  $A$  may be replaced by any abelian group with a commutative ring of operators for which the decomposition theorem is valid. It can then be seen easily that the proof given in this note remains valid without essential changes.

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