SOME INVARIANTS OF CERTAIN PAIRS
OF HYPERSURFACES

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Introduction. It is known \([8, 9]\) that if two surfaces in ordinary space have a common tangent plane at an ordinary point, then the ratio of their total curvatures at this point is a projective invariant, and the theorem holds true similarly for hyperspaces.\(^2\) In connection with this theorem and the investigation of Bouton \([2]\), Buzano \([3]\) and Bompiani \([1]\) have shown the existence of a projective invariant, together with metric and projective characterizations, determined by the neighborhood of the second order of two surfaces \(S, S^*\) at two ordinary points \(O, O^*\) in ordinary space under the conditions that the tangent planes of the surfaces \(S, S^*\) at the points \(O, O^*\) be distinct and have \(OO^*\) for the common line. Furthermore, the other case in which the tangent planes of the surfaces \(S, S^*\) at the points \(O, O^*\) are coincident has been considered in recent papers of the author \([6, 7]\).

It is the purpose of the present paper to generalize the results of the two cases mentioned above.

Let \(V_{n-1}, V^*_{n-1}\) be two hypersurfaces in a space \(S_n\) of \(n\) dimensions, and \(t_{n-1}, t^*_{n-1}\) the tangent hyperplanes of the hypersurfaces \(V_{n-1}, V^*_{n-1}\) at two ordinary points \(O, O^*\). For the subsequent discussion it is convenient to assume in Chapter I that the tangent hyperplanes \(t_{n-1}, t^*_{n-1}\) are coincident. We can \((§1)\), as in ordinary space, determine a projective invariant by the neighborhood of the second order of the hypersurfaces \(V_{n-1}, V^*_{n-1}\) at the points \(O, O^*\); and the projective and metric characterizations of this invariant are given in the next two sections.

Chapter II treats the case in which the tangent hyperplanes \(t_{n-1}, t^*_{n-1}\) are distinct, and the common tangent flat space \(t_{n-2}\) of \(t_{n-1}, t^*_{n-1}\) contains the line \(OO^*\). We first \((§4)\) show by analysis the existence of two projective invariants determined by the neighbor-
hood of the second order of the hypersurfaces $V_{n-1}$, $V^*_{n-1}$ at the points $O$, $O^*$; and then ($§§5, 6$) give them simple projective and metric characterizations. From the fact that one of the two invariants is reduced to 1 when the immersed space $S_n$ is of three dimensions, it follows that our result in this chapter stands actually for a generalization of that of Buzano and Bompiani.

CHAPTER I. TWO HYPERSURFACES WITH COMMON TANGENT HYPERPLANE AT TWO ORDINARY POINTS

1. Derivation of an invariant. Let $V_{n-1}$, $V^*_{n-1}$ be two hypersurfaces in a space $S_n$ of $n$ dimensions with common tangent hyperplane $t_{n-1}$ at two ordinary points $O$, $O^*$. Let $x_1, \ldots, x_{n+1}$ denote the homogeneous projective coordinates of a point in the space $S_n$. If we choose the points $O$, $O^*$ to be the vertices $(1, 0, \ldots, 0)$, $(0, \ldots, 0, 1, 0)$ of the system of reference, and the common tangent hyperplane $t_{n-1}$ to be the coordinate hyperplane $x_{n+1} = 0$ of the system, then the power series expansions of the hypersurfaces $V_{n-1}$, $V^*_{n-1}$ in the neighborhood of the points $O$, $O^*$ may be written in the form

\begin{align}
V_{n-1}: \quad & \frac{x_{n+1}}{x_1} = \sum_{i,k=2}^n l_{ik} \frac{x_i}{x_1} \frac{x_k}{x_1} + \cdots, \\
V^*_{n-1}: \quad & \frac{x_{n+1}}{x_n} = \sum_{i,k=1}^{n-1} m_{ik} \frac{x_i}{x_n} \frac{x_k}{x_n} + \cdots.
\end{align}

In order to find a projective invariant of the hypersurfaces $V_{n-1}$, $V^*_{n-1}$ at the points $O$, $O^*$, we have to consider the most general projective transformation of coordinates which shall leave the points $O$, $O^*$ and the hyperplane $t_{n-1}$ unchanged:

\begin{equation}
\begin{align}
& x_i = \sum_{r=1}^{n+1} a_{ir} x'_r \quad (i = 1, \ldots, n), \\
& x_{n+1} = a_{n+1,n+1} x'_{n+1},
\end{align}
\end{equation}

where

\begin{equation}
a_{21} = \cdots = a_{n1} = 0, \quad a_{1n} = \cdots = a_{n-1,n} = 0,
\end{equation}

\begin{equation}
D = \begin{vmatrix}
a_{22} & a_{23} & \cdots & a_{2,n-1} \\
a_{32} & a_{33} & \cdots & a_{3,n-1} \\
\vdots & \vdots & \ddots & \vdots \\
a_{n-1,2} & a_{n-1,3} & \cdots & a_{n-1,n-1}
\end{vmatrix} \neq 0.
\end{equation}

The effect of this transformation on equations (1), (2) is to produce
two other equations of the same form whose coefficients, indicated by accents, are given by the formulas

\[ a_{1i} a_{n+1,n+1} m'_{ik} = \sum_{r,s=2}^{n} a_{rs} a_{zrs} \text{ for } i, k = 2, \ldots, n, \]
\[ a_{nn} a_{n+1,n+1} m'_{k} = \sum_{r,s=1}^{n-1} a_{rs} a_{zrs} \text{ for } i, k = 1, \ldots, n-1. \]

From equations (4), (5), (6) it is easily seen that the determinants

\[
L = \begin{vmatrix} l_{22} & l_{23} & \cdots & l_{2n} \\ l_{32} & l_{33} & \cdots & l_{3n} \\ \vdots & \vdots & \ddots & \vdots \\ l_{n2} & l_{n3} & \cdots & l_{nn} \end{vmatrix}, \quad M = \begin{vmatrix} m_{11} & m_{12} & \cdots & m_{1,n-1} \\ m_{21} & m_{22} & \cdots & m_{2,n-1} \\ \vdots & \vdots & \ddots & \vdots \\ m_{n-1,1} & m_{n-1,2} & \cdots & m_{n-1,n-1} \end{vmatrix},
\]

and their transformed ones \( L', M' \) are connected by the relations

\[
\begin{align*}
\begin{bmatrix} a_{11} & a_{n+1,n+1} \\ a_{nn} & a_{n+1,n+1} \end{bmatrix} L' &= \begin{bmatrix} 2 \\ 2 \end{bmatrix} D^2 L, \\
\begin{bmatrix} a_{11} & a_{n+1,n+1} \\ a_{nn} & a_{n+1,n+1} \end{bmatrix} M' &= \begin{bmatrix} 2 \\ 2 \end{bmatrix} D^2 M.
\end{align*}
\]

Further elimination of \( a_{i+k} \) from equations (6), (7) shows immediately that the quantity

\[ I = \frac{L}{M} \left( \frac{m_{11}}{l_{nn}} \right)^{(n+1)/3} \]

is a projective invariant determined by the neighborhood of the second order of the hypersurfaces \( V_{n-1}, V^*_n \) at the points \( O, O^* \).

2. A projective characterization of the invariant \( I \). Let the polar spaces of the line \( OO^* \) with respect to the asymptotic hypercones of the hypersurfaces \( V_{n-1}, V^*_n \) at the points \( O, O^* \) be respectively denoted by \( t_{n-2}, t^*_{n-2} \), which determine a space \( t_{n-3} \) of \( n-3 \) dimensions in the common tangent hyperplane \( x_{n+1} = 0 \). If the \( n-2 \) vertices, other than \( O \) and \( O^* \), of the system of reference in the hyperplane \( x_{n+1} = 0 \) be chosen in the space \( t_{n-3} \), then the invariant \( I \) may be reduced to

\[ I = \frac{L_{nn}}{M_{11}} \left( \frac{m_{11}}{l_{nn}} \right)^{(n-2)/3}, \]

where \( L_{nn}, M_{11} \) are the minors of \( l_{nn}, m_{11} \) in the determinants \( L, M \) respectively.

For the purpose of finding a projective characterization of the in-
variant $I$ we first observe the space $S_8$ determined by the vertices $(1, 0, \cdots, 0), (0, \cdots, 0, 1, 0), (0, \cdots, 0, 1)$ and any one, say for instance $O_2(0, 1, 0, \cdots, 0)$, of the system of reference in the space $t_{n-3}$. The space $S_8$ intersects the hypersurfaces $V_{w-i}, V_{w-*i}$ in two surfaces $S, S^*$. Since the tangent planes of the surfaces $S, S^*$ at the points $O, O^*$ are coincident we have a projective invariant, denoted by $J$,

$$J = \frac{l_{22}}{m_{22}} \left( \frac{m_{11}}{l_{nn}} \right)^{1/8},$$

whose projective characterization has been obtained [6].

Let $Q (Q^*)$ be any quadric in the space $S_8$ which has $OO_2 (O^*O_2)$, $OO^* (OO^*)$ for generators and whose curve of intersection with the element of the second order of the surface $S (S^*)$ at the point $O (O^*)$ has a cusp at $O (O^*)$. If the cone projecting from the point $O_2$ the curve of intersection of the two quadrics $Q, Q^*$ be tangent to the common tangent plane $OO^*O_2$ along a line through the point $O_2$, then this line must be one of the lines (cf. [6])

$$x_n \pm (\pm 1)^{1/2} \left( \frac{m_{11}}{l_{nn}} \right)^{1/4} x_1 = 0,$$

$$x_2 = \cdots = x_{n-1} = x_{n+1} = 0.$$

We may now uniquely determine a point $P$ on the line $OO^*$ such that the cross ratio of the three points $O, O^*, P$, and the intersection of the line (11) with $OO^*$ is equal to $J^{1/4}$. On the other hand, the asymptotic hypercones of the hypersurfaces $V_{n-1}, V_{n-*1}$ at the points $O, O^*$ determine a pencil of hyperquadrics in the hyperplane $x_{n+1}=0$, among which there exist $n$ hypercones, two of them being the asymptotic hypercones. The line $OO^*$ intersects each of the other $n-2$ hypercones in a pair of points. Let $Q_i (i=1, \cdots, n-2)$ be any one of each pair of these points and $D_i$ the cross ratio of the four points $O, O^*, Q_i, P$ on the line $OO^*$, then we may easily show that the invariant $I$ can be expressed in terms of the $n-2$ cross ratios $D_1, D_2, \cdots, D_{n-2}$ as follows:

$$I = (\pm 1)^{n-2}(D_1D_2 \cdots D_{n-2})^2.$$

3. A metric characterization of the invariant $I$. It is deemed worth while to give in this section a simple metric characterization of the invariant $I$. For this purpose we choose an orthogonal Cartesian coordinate system in such a way that the point $O$ be the origin, the line $OO^*$ be the $X_{n-1}$-axis, and the common tangent hyperplane $t_{n-1}$ be the coordinate hyperplane $X_n=0$. Then the power series expan-
sions of the hypersurfaces \( V_{n-1}, V_{n-1}' \) in the neighborhood of the points \( O, O' \) may be put into the form

\[
V_{n-1}: \ X_n = \sum_{i,k=1}^{n-1} \lambda_{ik} X_i X_k + \cdots,
\]

\[
V_{n-1}': \ X_n = \sum_{i,k=1}^{n-2} \mu_{ik} X_i X_k + 2 \sum_{i=1}^{n-2} \mu_{i,n-1} X_i (X_{n-1} - h) + \mu_{n-1,n-1} (X_{n-1} - h)^2 + \cdots,
\]

where \( h \) is the distance between the points \( O, O' \).

Let \( y_0, y_1, \ldots, y_n \) be the homogeneous coordinates of a point defined by the formulas

\[
X_i = y_i / y_0 \quad (i = 1, \ldots, n),
\]

and let us consider the most general projective transformation of coordinates which shall leave the point \( O \) and the common tangent hyperplane \( t_{n-1} \) invariant, and change the point \( O' \) into the vertex \((0, \ldots, 0, 1, 0)\) of the new coordinate system:

\[
y_0 = \sum_{i=0}^{n} a_{0i} y_i,
\]

\[
y_i = \sum_{r=1}^{n} a_{ir} y_r' \quad (i = 1, \ldots, n - 1),
\]

\[
y_n = a_{nn} y'_n,
\]

where

\[
a_{1,n-1} = \cdots = a_{n-2,n-1} = 0, \quad a_{n-1,n-1} = h a_{0,n-1},
\]

\[
\Delta = \begin{vmatrix} a_{11} & a_{12} & \cdots & a_{1,n-2} \\ a_{21} & a_{22} & \cdots & a_{2,n-2} \\ \cdots & \cdots & \cdots & \cdots \\ a_{n-2,1} & a_{n-2,2} & \cdots & a_{n-2,n-2} \end{vmatrix} \neq 0.
\]

By transformations (15) and (16), equations (13), (14) shall be carried into two others of the form

\[
V_{n-1}: \ \frac{y'_n}{y'_0} = \sum_{i,k=1}^{n-1} p_{ik} \frac{y'_i}{y'_0} \frac{y'_k}{y'_0} + \cdots,
\]

\[
V_{n-1}': \ \frac{y'_n}{y'_{n-1}} = \sum_{i,k=0}^{n-2} q_{ik} \frac{y'_i}{y'_{n-1}} \frac{y'_k}{y'_{n-1}} + \cdots,
\]

where the coefficients \( p_{ik}, q_{ik} \) are given by the equations:
(21) \[ a_{00}a_{nn}p_{ik} = \sum_{r,s=1}^{n-1} a_{rs}a_{sh}\lambda_{rs} \quad (i, k = 1, \ldots, n - 1); \]
(22) \[ a_{nn}a_{0,n-1}q_{ik} = \sum_{r,s=0}^{n-2} a_{rs}a_{sh}\mu_{rs} \quad (i, k = 0, 1, \ldots, n - 2), \]
(23) \[ \alpha_{00} = -ha_{00}, \quad \alpha_{10} = 0, \quad \alpha_{0i} = a_{n-1,i} - ha_{0i}, \quad \alpha_{ri} = a_{ri}, \]
\[ \mu_{00} = \mu_{n-1,n-1}, \quad \mu_{0r} = \mu_{r0} = \mu_{n-1,r} = \mu_{r,n-1} \quad (i, r = 1, \ldots, n - 2). \]

Let
\[
\Phi = \begin{vmatrix}
\lambda_{11} & \lambda_{12} & \cdots & \lambda_{1,n-1} \\
\lambda_{21} & \lambda_{22} & \cdots & \lambda_{2,n-1} \\
\vdots & \vdots & \ddots & \vdots \\
\lambda_{n-1,1} & \lambda_{n-1,2} & \cdots & \lambda_{n-1,n-1}
\end{vmatrix}, \quad \Psi = \begin{vmatrix}
\mu_{11} & \mu_{12} & \cdots & \mu_{1,n-1} \\
\mu_{21} & \mu_{22} & \cdots & \mu_{2,n-1} \\
\vdots & \vdots & \ddots & \vdots \\
\mu_{n-1,1} & \mu_{n-1,2} & \cdots & \mu_{n-1,n-1}
\end{vmatrix},
\]
\[
P = \begin{vmatrix}
p_{11} & p_{12} & \cdots & p_{1,n-1} \\
p_{21} & p_{22} & \cdots & p_{2,n-1} \\
\vdots & \vdots & \ddots & \vdots \\
p_{n-1,1} & p_{n-1,2} & \cdots & p_{n-1,n-1}
\end{vmatrix}, \quad Q = \begin{vmatrix}
q_{00} & q_{01} & \cdots & q_{0,n-2} \\
q_{10} & q_{11} & \cdots & q_{1,n-2} \\
\vdots & \vdots & \ddots & \vdots \\
q_{n-2,0} & q_{n-2,1} & \cdots & q_{n-2,n-2}
\end{vmatrix},
\]

then from equations (17), (18), (21), (22), (23) we obtain
\[
(24) \quad a_{n-1,n-1}a_{00}a_{nn}P = \frac{2}{a_{n-1,n-1}\Delta^2} \Phi, \quad a_{n-1,n-1}a_{00}a_{nn}Q = \frac{2}{a_{n-1,n-1}\Delta^2} \Psi.
\]

Making use of the result obtained in §1 and observing equations (19), (20) we see that the projective invariant \( I \) associated with the hypersurfaces \( V_{n-1}, V^*_{n-1} \) at the points \( O, O^* \) is
\[
(25) \quad I = P \left( \frac{q_{00}}{p_{n-1,n-1}} \right)^{(n+1)/4}.
\]

Furthermore, substituting (21), (22), (24) in (25) and reducing by equations (17) it follows that the invariant \( I \) now takes the form
\[
(26) \quad I = \frac{\Phi}{\Psi} \left( \frac{\mu_{n-1,n-1}}{\lambda_{n-1,n-1}} \right)^{(n+1)/4}.
\]

Let \( K, K^* \) be the curvatures of the hypersurfaces \( V_{n-1}, V^*_{n-1} \) at the points \( O, O^* \); and \( R, R^* \) the curvatures at the points \( O, O^* \) of the plane sections of the hypersurfaces \( V_{n-1}, V^*_{n-1} \) made by the plane of the line \( OO^* \) and the normal to the common tangent hyperplane \( t_{n-1} \) at any point on the line \( OO^* \). By a known formula it is easy to
demonstrate that
\[ K/K^* = \Phi/\Psi, \quad R/R^* = \lambda_{n-1,n-1}/\mu_{n-1,n-1}, \]
and therefore that
\[ I = \frac{K}{K^*} \left( \frac{R^*}{R} \right)^{(n+1)/3}. \]

Hence we have the following theorem.

**Theorem.** Let \( V_{n-1}, V^*_{n-1} \) be two hypersurfaces in a space \( S_n \) of \( n \) dimensions having a common tangent hyperplane \( t_{n-1} \) at two ordinary point \( O, O^*; K, K^* \) the curvatures of the hypersurfaces \( V_{n-1}, V^*_{n-1} \) at the points \( O, O^*; \) and \( R, R^* \) the curvatures at the points \( O, O^* \) of the plane sections of the hypersurfaces \( V_{n-1}, V^*_{n-1} \) made by the plane of the line \( OO^* \) and the normal to the common tangent hyperplane \( t_{n-1} \) at any point on the line \( OO^* \). Then \( (K/K^*)(R^*/R)^{(n+1)/3} \) is a projective invariant associated with the hypersurfaces \( V_{n-1}, V^*_{n-1} \) at the points \( O, O^* \).

**Chapter II. Two Hypersurfaces with Distinct Tangent Hyperplanes at Two Ordinary Points**

4. Derivation of invariants. Let \( V_{n-1}, V^*_{n-1} \) be two hypersurfaces in a space \( S_n \) of \( n \) dimensions such that the tangent hyperplanes \( t_{n-1}, t^*_{n-1} \) at two ordinary points \( O, O^* \) are distinct, and the common tangent flat space \( t_{n-2} \) of \( t_{n-1}, t^*_{n-1} \) contains the line \( OO^* \). If we choose the points \( O, O^* \) to be the vertices \((0, 1, 0, \ldots, 0), (0, \ldots, 0, 1, 0)\) of a homogeneous projective coordinate system of reference, and the tangent hyperplanes \( t_{n-1}, t^*_{n-1} \) to be the coordinate hyperplanes \( x_1 = 0, x_{n+1} = 0 \) respectively, then the power series expansions of the hypersurfaces \( V_{n-1}, V^*_{n-1} \) in the neighborhood of the points \( O, O^* \) may be written in the form

\[ V_{n-1}: \quad \frac{x_1}{x_2} = \sum_{i,k=1}^{n+1} l_{ik} \frac{x_i}{x_2} + \cdots, \]
\[ V^*_{n-1}: \quad \frac{x_{n+1}}{x_n} = \sum_{i,k=1}^{n-1} m_{ik} \frac{x_i}{x_n} + \cdots. \]

Considering the most general projective transformation of coordinates which shall leave the points \( O, O^* \) and the hyperplanes \( t_{n-1}, t^*_{n-1} \) unchanged, we may easily show as in §1 that the quantities

\[ I = \frac{L M l_{n,n+1} m_{22}}{L_{n+1,n+1} M_{11}}, \quad J = \left( \frac{M}{L} \right)^{n-3} \left( \frac{L_{n+1,n+1} m_{22}}{M_{11} l_{n,n+1}} \right)^{n+1}. \]
are projective invariants determined by the neighborhood of the second order of the hypersurfaces $V_{n-1}$, $V_{n-1}^*$ at the points $O, O^*$, where $L_{n+1,n+1}$, $M_{11}$ are respectively the minors of $l_{n+1,n+1}$, $m_{11}$ in the determinants

$$L = \begin{vmatrix} l_{33} & l_{34} & \cdots & l_{3,n+1} \\ l_{43} & l_{44} & \cdots & l_{4,n+1} \\ \vdots & \vdots & \ddots & \vdots \\ l_{n+1,3} & l_{n+1,4} & \cdots & l_{n+1,n+1} \end{vmatrix}, \quad M = \begin{vmatrix} m_{11} & m_{12} & \cdots & m_{1,n-1} \\ m_{21} & m_{22} & \cdots & m_{2,n-1} \\ \vdots & \vdots & \ddots & \vdots \\ m_{n-1,1} & m_{n-1,2} & \cdots & m_{n-1,n-1} \end{vmatrix},$$

and $L', M', L'_{n+1,n+1}, M'_{11}$ are denoted by similar expressions.

5. Projective characterizations of the invariants $I, J$. By suitable choice of the system of reference the invariants $I, J$ of equations (31) can be simplified. In fact, if we choose $n-1$ vertices of the system in the common tangent flat space $t_{n-2}$, and the other two $O_{n+1}(0, \cdots, 0, 1), O_1(1, 0, \cdots, 0)$ respectively on the polars $t, t^*$ of the flat space $t_{n-2}$ with respect to the asymptotic hypercones of the hypersurfaces $V_{n-1}, V_{n-1}^*$ at the points $O, O^*$, the invariants $I, J$ then take the simple form

$$I = l_{n} l_{n+1,n+1} m_{11} m_{22},$$

$$J = \left( \frac{l_{n+1,n+1}}{M_{11}} \right)^4 \left( \frac{m_{11}}{l_{n+1,n+1}} \right)^{n-3} \left( \frac{m_{22}}{l_{nn}} \right)^{n+1}.$$  

It should be noticed that the invariant $J$ is reduced to 1 as $n = 3$.

The polars $t, t^*$ determine a space $S_3$, which intersects the hypersurfaces $V_{n-1}, V_{n-1}^*$ in two surfaces $S, S^*$. These two surfaces $S, S^*$ are evidently in the class considered by Buzano and Bompiani, and the corresponding invariant may be easily found from Bompiani's note [1] to coincide just with the invariant $I$. Thus we reach the conclusion:

The invariant $I$ associated with the hypersurfaces $V_{n-1}, V_{n-1}^*$ at the points $O, O^*$ is the invariant of Buzano at the points $O, O^*$ of the surfaces $S, S^*$ in which the hypersurfaces $V_{n-1}, V_{n-1}^*$ are intersected by the space $S_3$ determined by the polars $t, t^*$.

To characterize projectively the other invariant $J$ we consider any hyperplane $\pi_\alpha$ through the common tangent flat space $t_{n-2}$:

$$x_{n+1} = \alpha x_1 \quad (\alpha \neq 0),$$

which intersects the hypersurfaces $V_{n-1}, V_{n-1}^*$ in two hypersurfaces $V_{n-2}, V_{n-2}^*$ of $n-2$ dimensions. Since these two hypersurfaces $V_{n-2}, V_{n-2}^*$ have a common tangent hyperplane at the points $O, O^*$ we may
determine an invariant, denoted by $I_a$, as in §1:

$$I = \alpha^{n-2/3} \frac{L_{n+1,n+1}}{M_{11}} \left( \frac{m_{33}}{l_{nn}} \right)^{n/8}. \quad (34)$$

On the other hand, it is useful to consider the hypercones $C, C^*$ projecting respectively from the vertices $O_i(1, 0, \cdots, 0), O_{n+1}(0, \cdots, 0, 1)$ the asymptotic hypercones at the points $O, O^*$ of the hypersurfaces $V_n, V_n^*$. These two hypercones $C, C^*$ determine a pencil of hyperquadrics in the space $S_n$, among which there exist $n-1$ hypercones, two of them being $C, C^*$. The line $O_iO_{n+1}$ intersects each of the other $n-3$ hypercones in a pair of points. Let $Q_i, (i=1, \cdots, n-3)$ be any one of each pair of these points, $P$ the point of intersection of the line $O_iO_{n+1}$ with the hyperplane $\pi_i$, and $D_i$ the cross ratio of the four points $O_i, O_{n+1}, Q_i, P$ on the line $O_iO_{n+1}$; then it follows that the invariant $J$ can be expressed in terms of the invariant $I_a$ and the $n-3$ cross ratios $D_1, D_2, \cdots, D_{n-3}$ as follows:

$$J = I_a(D_1D_2 \cdots D_{n-3})^2. \quad (35)$$

6. Metric characterizations of the invariants $I, J$. For the purpose of finding simple metric characterizations of the invariants $I, J$, we choose an orthogonal Cartesian coordinate system in such a way that the point $O$ is the origin, the line $OO^*$ is the $X_n$-axis, and the tangent hyperplane $t_{n-1}$ is the coordinate hyperplane $X_1=0$. Then the power series expansions of the hypersurfaces $V_n, V_n^*$ in the neighborhood of the points $O, O^*$ may be put into the form

$$V_n: \quad X_1 = \sum_{i,k=2}^{n} \lambda_{ik}X_iX_k + \cdots, \quad (36)$$

$$V_n^*: \quad X_n = \mu X_1 + \sum_{i,k=1}^{n-2} \mu_{ik}X_iX_k + 2\sum_{i=1}^{n-2} \mu_{i,n-1}X_i(X_{n-1} - h) + \mu_{n-1,n-1}(X_{n-1} - h)^2 + \cdots, \quad (37)$$

where $h$ is the distance between the points $O, O^*$, and $\mu = \cot \omega$, $\omega$ being the angle of the tangent hyperplanes $t_{n-1}, t_{n-1}^*$. In order to express the two invariants $I, J$ in terms of the coefficients of expansions (36), (37) we have first as in §3 to consider the homogeneous coordinates $y_0, y_1, \cdots, y_n$ of a point defined by formulas (15) and the most general projective transformation of coordinates, which shall leave the point $O$ and the tangent hyperplane $t_{n-1}$ invariant and carry the point $O^*$ and the tangent hyperplane $t_{n-1}^*$ into the vertex $(0, \cdots, 0, 1, 0)$ and the coordinate hyperplane.
\( y_i' = 0 \) of the new coordinate system respectively. An easy calculation, which shall be omitted here, suffices to demonstrate the result as follows:

\[(38) \quad I = \frac{\Phi \Psi \lambda_{n-1,n-1} \mu_{n-1,n-1}}{\Phi_{nn} \Psi_{11}}, \quad J = \left( \frac{\Psi}{\Phi} \right)^{n-8} \left( \frac{\Phi_{nn} \mu_{n-1,n-1}}{\Psi_{11} \lambda_{n-1,n-1}} \right)^{n+1}, \]

where \( \Phi_{nn}, \Psi_{11} \) denote respectively the minors of \( \lambda_{nn}, \mu_{11} \) in the determinants

\[
\Phi = \begin{vmatrix} 
\lambda_{22} & \cdots & \lambda_{2n} \\
\lambda_{22} & \cdots & \lambda_{3n} \\
\cdots & \cdots & \cdots \\
\lambda_{n2} & \cdots & \lambda_{nn} 
\end{vmatrix}, \quad \Psi = \begin{vmatrix} 
\mu_{11} & \cdots & \mu_{1,n-1} \\
\mu_{21} & \cdots & \mu_{2,n-1} \\
\cdots & \cdots & \cdots \\
\mu_{n-1,1} & \cdots & \mu_{n-1,n-1} 
\end{vmatrix}.
\]

Finally, we shall make use of the normals \( ON, ON^* \) at the point \( O \) of the common tangent flat space \( t_{n-2} \) in the tangent hyperplanes \( t_{n-1}, t_{n-1}^* \). Let \( K_2, K_*^* \) be respectively the curvatures at the points \( O, O^* \) of the plane sections of the hypersurfaces \( V_{n-1}, V_{n-1}^* \) made by the planes \( OO^* N^*, OO^* N \). Further, let \( K_n, K_n^* \) be the curvatures of the hypersurfaces \( V_{n-1}, V_{n-1}^* \) at the points \( O, O^* \); and \( K_{n-1}, K_{n-1}^* \) the curvatures at the points \( O, O^* \) of the hypersurfaces \( V_{n-2}, V_{n-2}^* \) of \( n-2 \) dimensions in which the tangent hyperplanes \( t_{n-1}^*, t_{n-1} \) intersect the hypersurfaces \( V_{n-1}, V_{n-1}^* \) respectively. Then

\[
K_n = 2^{n-1} \Phi, \quad K_n^* = 2^{n-1} (1 + \mu^2)^{-(n+1)/2} \Phi,
\]

\[
(39) \quad K_{n-1} = 2^{n-2} (1 + \mu^2)^{(n-2)/2} \Phi_{nn}, \quad K_{n-1}^* = 2^{n-2} \Phi_{11},
K_2 = 2(1 + \mu^2)^{1/2} \lambda_{n-1,n-1}, \quad K_2^* = 2\mu_{n-1,n-1},
\]

and hence we arrive at the following metric characterizations of the invariants \( I, J \):

\[
(40) \quad I = \frac{h^4}{16} \frac{K_n K_n^* K_2 K_2^*}{K_{n-1} K_{n-1}^* \sin^2 (\omega-1) \omega}, \quad J = \left( \frac{K_n^*}{K_n} \right)^{n-3} \left( \frac{K_{n-1} K_2^*}{K_{n-1} K_2} \right)^{n+1}.
\]

**Bibliography**


7. ———, *On a projective invariant of a certain pair of surfaces*, to be published elsewhere.


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