THE STRONG SUMMABILITY OF DOUBLE FOURIER SERIES

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1. Introduction. Corresponding to the well known theorem of Fejér-Lebesgue, we have for the double Fourier series the following proposition:

If \( f \log^+|f| \) is Lebesgue integrable on the square \((-\pi \leq x \leq \pi, -\pi \leq y \leq \pi)\), then the Fejér mean \( \sigma_{m,n}(x, y) \) of \( f(x, y) \) tends to \( f(x, y) \) almost everywhere as \( m \) and \( n \) independently increase indefinitely. Moreover, for every increasing function \( \phi(t) \) satisfying the conditions

\[
\phi(0) = 0, \quad \lim \inf_{t \to \infty} \frac{\phi(t)}{t \log t} = 0,
\]

there is a function \( f(x, y) \) such that \( \phi(|f|) \) is integrable and that \( \sigma_{m,n}(x, y) \) does not converge almost everywhere.\(^1\)

The latter half of this theorem shows that the analogue, in double Fourier series, of the Fejér-Lebesgue theorem is not a trivial extension of that of a function of a single variable.

The purpose of the present note is to discuss the strong summability\(^2\) of double Fourier series. A double series \( \sum a_{m,n} \) is said to be strongly summable with the positive index \( k \) if there exists a constant \( s \) such that the expression

\[
(1.1) \quad \frac{1}{(m+1)(n+1)} \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} | s_{m,n} - s |^k
\]

has the double limit zero as \( m \) and \( n \) increase without limit, where

\[
s_{m,n} = \sum_{\mu=0}^{m} \sum_{\nu=0}^{n} a_{\mu\nu}.
\]

It is easily seen from Hölder's equality that the summability says more for larger \( k \).

Suppose now that \( f(x, y) \) is integrable in the Lebesgue sense over

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\(^1\) B. Jessen, J. Marcinkiewicz and A. Zygmund [5]. The first example of a function \( f(x, y) \in L \) with Fejér mean divergent everywhere was given by A. Zygmund; see S. Saks [8]. Numbers in brackets refer to the Bibliography at the end of the paper.

\(^2\) A notion first introduced in Fourier series by G. H. Hardy and J. E. Littlewood [1]. For subsequent researches, see Hardy and Littlewood [2, 3], J. Marcinkiewicz [6] and A. Zygmund [12].

700
the square $Q (-\pi, -\pi; \pi, \pi)$ and is doubly periodic with period $2\pi$ in each variable. The Fourier series of $f(x, y)$ is

$$\sum_{m,n=0}^{\infty} \lambda_{m,n} \left[ a_{m,n} \cos mx \cos ny + b_{m,n} \sin mx \cos ny + c_{m,n} \cos mx \sin ny + d_{m,n} \sin mx \sin ny \right],$$

where

$$\lambda_{m,n} = \begin{cases} 1/4 & \text{for } m = n = 0; \\ 1/2 & \text{for } m = 0, n > 0 \text{ or } m > 0, n = 0; \\ 0 & \text{for } m > 0, n > 0; \end{cases}$$

and

$$a_{m,n} = \frac{1}{\pi^2} \int_{Q} \int f(x, y) \cos mx \cos ny \, dx \, dy,$$

and so on.

On writing

$$4\phi(u, v) \equiv \phi_{x,y}(u, v) = f(x + u, y + v) + f(x + u, y - v) + f(x - u, y + v) + f(x - u, y - v) - 4s,$$

and

$$\Phi_{x,y}(u, v) = \int_{0}^{u} \int_{0}^{v} |\phi(\xi, \eta)|^p d\xi d\eta \quad (p \geq 1)$$

the theorems obtained in this paper are as follows:

**Theorem I.** If $f(x, y) \in L^p$, $p > 1$, then the double Fourier series (1.2) is strongly summable to $s$ for every positive index $k$ whenever

$$\Phi_{x,y}(u, v) = o(1).$$

**Theorem II.** If $f(x, y) \in L^p$, $p > 1$, then the Fourier series of $f(x, y)$ is strongly summable almost everywhere to $f(x, y)$ for every positive index $k$.

The question whether the hypothesis in Theorem II may be replaced by $f \log^+ |f| \in L$ is unsettled in this note. Corresponding questions in Fourier series of a single variable have been answered affirmatively by Marcinkiewicz [6] and Zygmund [12]. Indeed, the theorem holds under the weaker hypothesis $f \in L$. We content ourselves with establishing the following theorem.

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*We use the symbol $o(1)$ to denote a function of $u$ and $v$ such that $\lim_{u,v \to 0} o(uv)/uv = 0$. 
THEOREM III. If \( f(x, y) \) log+ \( |f(x, y)| \in L \) and \( \sigma_{m,n}(x, y) \) denotes the \((m, n)\)th Fejér sum of the Fourier series of \( f(x, y) \), then the relation
\[
\lim_{m,n \to \infty} \frac{1}{(m + 1)(n + 1)} \sum_{\mu=0}^{m} \sum_{\nu=0}^{n} |\sigma_{\mu,\nu}(x, y) - f(x, y)|^k = 0
\]
holds true almost everywhere, where \( k > 0 \).

2. Lemmas. Before proving our theorems, we prove a number of lemmas:

**Lemma 1.** If \( f(x, y) \in L^p, p > 1 \), then
\[
\lim_{h,k \to 0} \frac{1}{h k} \int_{x_0}^{x_0+h} \int_{y_0}^{y_0+k} f(x, y) \, dx \, dy = f(x_0, y_0)
\]
at almost every point \((x_0, y_0)\).

This theorem is due to Zygmund [11]. Compare also [5] and [9].

**Lemma 2.** If \( f(x, y) \in L^p, p > 1 \), then at almost every point \((x, y)\),
\[
\int_0^h \int_0^k |f(x \pm u, y \pm v) - f(x, y)|^p \, du \, dv = o(hk)
\]
as \( h, k \to 0 \).

**Proof.** Let \( \alpha \) be a rational number, and \( E_\alpha \) the set of points \((x, y)\) such that
\[
\frac{1}{hk} \int_0^h \int_0^k |f(x \pm u, y \pm v) - \alpha|^p \, du \, dv
\]
does not tend to \( |f(x, y) - \alpha|^p \) as \( h, k \to 0 \). In virtue of Lemma 1, \( E_\alpha \) is of measure zero, and so also is the sum \( E \) of all \( E_\alpha \). Let \((x, y)\) be not a point of \( E \) and let \( \beta \) be a rational number, then, by Minkowski’s inequality,
\[
\left\{ \frac{1}{hk} \int_0^h \int_0^k |f(x \pm u, y \pm v) - f(x, y)|^p \, du \, dv \right\}^{1/p}
\leq \left\{ \frac{1}{hk} \int_0^h \int_0^k |f(x \pm u, y \pm v) - \beta|^p \, du \, dv \right\}^{1/p}
\]
\[+ \left\{ \frac{1}{hk} \int_0^h \int_0^k |\beta - f(x, y)|^p \, du \, dv \right\}^{1/p},\]
which tends to \( 2|f(x, y) - \beta| \) as \( h, k \to 0 \). As \( \beta \to f(x, y) \), the result follows.
Lemma 3. Let \( f(x, y) \in L^p, 1 < p \leq 2, 1/p + 1/q = 1 \), and let the Fourier series of \( f(x, y) \) be given in the complex form:

\[
f(x, y) \sim \sum_{\mu=-\infty}^{\infty} \sum_{\nu=-\infty}^{\infty} c_{\mu, \nu} e^{i(x\mu + y\nu)},
\]

then

\[
\left\{ \sum_{\mu=-\infty}^{\infty} \sum_{\nu=-\infty}^{\infty} |c_{\mu, \nu}|^q \right\}^{1/q} \leq \left\{ \frac{1}{4\pi^2} \int_{-\pi}^{\pi} \int_{-\pi}^{\pi} |f(x, y)|^p \, dx \, dy \right\}^{1/p}.
\]

This is a double series analogue of the Young-Hausdorff theorem, and may be proved by the method of M. Riesz \( ^4 \) with an obvious modification.

We also require the following formula of integration by parts:

\[
\int_{a_1}^{a_2} du \int_{b_1}^{b_2} \rho \psi'(u) \psi''(v) \, du \, dv
\]

\[
= \rho_1(a_2, b_2) \psi(a_2, b_2) - \int_{a_1}^{a_2} \rho_1(u, b_2) \psi_u(u, b_2) \, du
\]

\[
- \int_{b_1}^{b_2} \rho_1(a_2, v) \psi_v(a_2, v) \, dv + \int_{a_1}^{a_2} du \int_{b_1}^{b_2} \rho_3 \psi_u \, dv,
\]

where

\[
\psi(u, v) = \psi'(u) \psi''(v), \quad \rho_1(u, v) = \int_{a_1}^{u} du \int_{b_1}^{v} \rho(\sigma, t) \, dt.
\]

This formula is valid if \( \rho \) is integrable on \((a_1, b_1; a_2, b_2)\), \( \psi' \) is absolutely continuous on \((a_1, a_2)\), and \( \psi'' \) is absolutely continuous on \((b_1, b_2)\).

3. Proof of Theorem I. Without loss of generality, we may assume that \( x = 0, y = 0 \). So that

\[
s_{m,n} = \frac{1}{\pi^2} \int_0^\pi \int_0^\pi f(u, v) \frac{\sin (m + 1/2)u}{\sin u/2} \frac{\sin (n + 1/2)v}{\sin v/2} \, du \, dv.
\]

We have to deduce

\[
\sum_{\mu=0}^{m} \sum_{\nu=0}^{n} |s_{\mu, \nu} - s|^k = o(uv)
\]

from (1.3).

Write

\( ^4 \) M. Riesz [7], see also A. Zygmund [13].
\[\pi^2(s_\mu, s_\nu - s) = \int_0^\pi \int_0^\pi \phi(u, v) \frac{\sin(m + 1/2)u}{\sin u/2} \frac{\sin(n + 1/2)v}{\sin v/2} \, du \, dv\]

\[= \int_0^\pi \int_0^\pi \phi(u, v) \left( \sin \mu u \cot \frac{u}{2} \sin \nu v + \sin \mu u \cot \frac{u}{2} \cos \nu v + \cos \mu u \sin \nu v \cot \frac{v}{2} + \cos \mu u \cos \nu v \right) \, du \, dv\]

\[= I_1(\mu, \nu) + I_2(\mu, \nu) + I_3(\mu, \nu) + I_4(\mu, \nu),\]

and for \(0 < \mu \leq m, \, 0 < \nu \leq n,\)

\[I_i(\mu, \nu) = \int_0^{m-1} \int_0^{n-1} + \int_0^{m-1} \int_{n-1}^{\pi} + \int_{m-1}^{\pi} \int_0^{n-1} + \int_{m-1}^{\pi} \int_{n-1}^{\pi}\]

\[= I_{1i}(\mu, \nu; m, n) + I_{2i}(\mu, \nu; m, n) + I_{3i}(\mu, \nu; m, n),\]

where \(i = 1, 2, 3.\) For brevity, we also write \(I_{\mu\nu}(\mu, \nu)\) for \(I_{\mu\nu}(\mu, \nu; m, n).\) Accordingly,

\[\pi^2(s_\mu, s_\nu - s) = \sum_{i=1}^3 \sum_{j=1}^4 I_{ij}(\mu, \nu) + I_4(\mu, \nu).\]

It follows from Minkowski's inequality that

\[\pi^2 \left\{ \sum_{\mu=0}^m \sum_{\nu=0}^n |s_\mu, s_\nu - s|^k \right\}^{1/k} \leq \sum_{i=1}^3 \sum_{j=1}^4 \left\{ \sum_{\mu=0}^m \sum_{\nu=0}^n |I_{ij}(\mu, \nu)|^k \right\}^{1/k} + \left\{ \sum_{\mu=0}^m \sum_{\nu=0}^n |I_4(\mu, \nu)|^k \right\}^{1/k}.\]

In the first place, by the analogue of the Riemann-Lebesgue theorem, \(I_4(\mu, \nu)\) tends to zero as \(\mu, \nu \to \infty.\) Hence

\[\sum_{\mu=0}^m \sum_{\nu=0}^n |I_4(\mu, \nu)|^k \leq o(mn^{1/k}).\]

Secondly, let us consider the integrals \(I_{11}, I_{21}\) and \(I_{31}.\) Write

\[K(u, v) = K(u, v; \mu, \nu) = \sin \mu u \cot \frac{u}{2} \sin \nu v \cot \frac{v}{2},\]

then for \(0 < u \leq \pi\) and \(0 < v \leq \pi\) there is a constant \(A\) such that

\[uv \max (|K|, \mu^{-1} |K_u|, \nu^{-1} |K_v|, \mu^{-1} \nu^{-1} |K_{uv}|) \leq A.\]

We also write

\[^5\text{W. H. Young [10, p. 138].}\]
\[ \Phi(u, v) = \int_0^u \int_0^v \phi(\xi, \eta) d\xi d\eta, \]

which is \( o(uv) \) by (1.3). Then on applying (2.2),

\[ I_{11}(\mu, \nu) = \int_0^{m-1} \int_0^{n-1} \phi(u, v) K(u, v; \mu, \nu) dudv \]
\[ = \Phi(m^{-1}, n^{-1}) K(m^{-1}, n^{-1}) \]
\[ - \int_0^{m-1} \Phi(u, n^{-1}) K_u(u, n^{-1}) du \]
\[ - \int_0^{n-1} \Phi(m^{-1}, v) K_v(m^{-1}, v) dv \]
\[ + \int_0^{m-1} du \int_0^{n-1} \Phi(u, v) K_{uv} dv. \]

(3.5)

Since \( 0 < \mu \leq m, 0 < \nu \leq n \), it is easily seen from (3.4) and (3.5) that

(3.6) \[ I_{11}(\mu, \nu) = o(1). \]

In a similar manner, we can prove \( I_{31}(\mu, \nu) = o(1), I_{31}(\mu, \nu) = o(1) \). Hence we obtain

(3.7) \[ \left\{ \sum_{\mu=0}^{m} \sum_{\nu=0}^{n} \left| I_{i1}(\mu, \nu) \right|^{k} \right\}^{1/k} = o(mn^{1/k}) \quad (i = 1, 2, 3). \]

Thirdly, we consider the integrals \( I_{14}, I_{24} \) and \( I_{34} \). We have

\[ I_{14}(\mu, \nu) = \int_{m-1}^{\nu} \int_{n-1}^{\nu} \phi(u, v) \sin \mu u \cot \frac{u}{2} \sin \nu v \cot \frac{v}{2} dudv \]
\[ = \int_{m-1}^{\nu} \sin \mu u \cot \frac{u}{2} du \]
\[ \cdot \int_{n-1}^{\nu} \cot \frac{v}{2} \left( \frac{\partial}{\partial \nu} \int_0^{\nu} \sin \nu y \phi(u, y) dy \right) dv \]
\[ = \int_{m-1}^{\nu} \sin \mu u \cot \frac{u}{2} du \left( - \cot \frac{1}{2n} \int_0^{n-1} \sin \nu y \phi(u, y) dy \right) \]
\[ + \frac{1}{2} \int_{n-1}^{\nu} \csc^2 \frac{v}{2} dv \int_0^{\nu} \sin \nu y \phi(u, y) dy \]
\[ = I'_{14} + I''_{14}, \]

say, where \( I'_{14} \) is equal to
Let $c_{\mu,v}(\alpha, \beta)$ denote the $(\mu, v)$th Fourier coefficient of the odd-odd function $\chi(x, y)$ which is equal to $\phi(x, y)$ in the rectangle $(0, \alpha; 0, \beta)$ and to zero elsewhere. Then we may write

$$I_{14}' = \frac{\pi^2}{8} \cot \frac{1}{2m} \cot \frac{1}{2n} c_{\mu,v} \left( \frac{1}{m}, \frac{1}{n} \right)$$

(3.9)

and $I_{14}''$ may be written as

$$\frac{1}{2} \int_{m-1}^{x} \cot \frac{u}{2} \, du$$

$$\cdot \int_{n-1}^{v} \csc^2 \frac{v}{2} \left( \frac{d}{du} \int_{0}^{u} \int_{0}^{v} \sin \mu x \sin \nu y \phi(x, y) \, dx \, dy \right) \, dv$$

(3.10)

It follows from (3.8), (3.9) and (3.10) that

$$\left( \sum_{\mu=0}^{m} \sum_{v=0}^{n} \left| I_{14}(\mu, v) \right|^k \right)^{1/k} \leq A \cot \frac{1}{2m} \cot \frac{1}{2n} \left( \sum_{\mu=0}^{m} \sum_{v=0}^{n} \left| c_{\mu,v} \left( \frac{1}{m}, \frac{1}{n} \right) \right|^k \right)^{1/k}$$

$$+ A \cot \frac{1}{2n} \int_{m-1}^{x} \csc^2 \frac{u}{2} \left( \sum_{\mu=0}^{m} \sum_{v=0}^{n} \left| c_{\mu,v} \left( u, \frac{1}{n} \right) \right|^k \right)^{1/k} \, du$$
\[
+ A \cot \frac{1}{2m} \int_0^\pi \csc^2 \frac{v}{2} \left( \sum_{\mu=0}^m \sum_{\nu=0}^n \left| c_{\mu,\nu} \left( \frac{1}{m}, \nu \right) \right|^k \right)^{1/k} dv \\
+ A \cot \frac{1}{2m} \cot \frac{1}{2n} \int_0^\pi \int_0^\pi \csc^2 \frac{u}{2} \csc^2 \frac{v}{2} \\
\cdot \left( \sum_{\mu=0}^m \sum_{\nu=0}^n \left| c_{\mu,\nu}(u, v) \right|^k \right)^{1/k} \, du \, dv.
\]

Now we assume, without loss of generality, that \(k > 2, \frac{k'}{k} = \frac{k}{k-1} < \rho\), so that by Lemma 3,
\[
\left( \sum_{\mu=0}^m \sum_{\nu=0}^n \left| c_{\mu,\nu}(u, v) \right|^k \right)^{1/k} \leq \left( \frac{1}{4\pi^2} \int_{-\pi}^{\pi} \int_{-\pi}^{\pi} \left| \chi(x, y) \right|^{k'} dxdy \right)^{1/k'} \\
= \left( \frac{1}{4\pi^2} \int_{-\pi}^{\pi} \int_{-\pi}^{\pi} \left| \phi(x, y) \right|^{k'} dxdy \right)^{1/k'} \\
= o(mn)^{1/k'},
\]
since the condition (1.3) is satisfied a fortiori when \(\rho\) is replaced by the smaller index \(k'\). Therefore
\[
\left( \sum_{\mu=0}^m \sum_{\nu=0}^n \left| I_{14}(\mu, \nu) \right|^{1/k} \right)^{1/k} \leq Amn(mn)^{-1/k'} + An \int_{m-1}^{\pi} \left( \frac{u}{n} \right)^{1/k'} \frac{du}{u^2} \\
+ Am \int_{n-1}^{\pi} \left( \frac{v}{m} \right)^{1/k'} \frac{dv}{v^2} \\
+ A \int_{m-1}^{\pi} \int_{n-1}^{\pi} \frac{(uv)^{1/k'}}{u^2v^2} \, dudv = o(mn)^{1/k}.
\]
The integral \(I_{24}(\mu, \nu)\) is equal to
\[
\int_{m-1}^{\pi} \cot \frac{u}{2} \left( \frac{d}{du} \int_0^u dx \int_{n-1}^{\pi} \phi(x, y) \sin \mu x \cos \nu y dy \right) du \\
= \frac{\pi^2}{4} \cot \frac{1}{2m} \left[ c'_{\mu, \nu} \left( \frac{1}{m}, \pi \right) - c'_{\mu, \nu} \left( \frac{1}{m}, \frac{1}{n} \right) \right] \\
- \frac{\pi^2}{8} \int_{m-1}^{\pi} \csc^2 \frac{u}{2} \left[ c'_{\mu, \nu}(u, \pi) - c'_{\mu, \nu}(u, \frac{1}{n}) \right] du,
\]
where the \(c'_{\mu, \nu}(\alpha, \beta) (\mu, \nu = 0, 1, 2, \cdots)\) denote the Fourier coefficients of the odd-even function \(\chi'(x, y)\) which is equal to \(\phi(x, y)\) in the rectangle \((0, \alpha; 0, \beta)\) and to zero elsewhere. In virtue of Minkowski's inequality and Lemma 3, it is easily seen that
The integral $I_{24}$ can be treated in the same manner as $I_{14}$. We omit the details. Collecting the above results, we obtain

\[
\left( \sum_{\mu=0}^{n} \sum_{\nu=0}^{n} |I_{24}(\mu, \nu)|^k \right)^{1/k} = o(mn)^{1/k},
\]

(3.11)

Fourthly, we estimate the integrals $I_{12}$, $I_{22}$ and $I_{32}$. We have

\[
I_{12}(\mu, \nu) = \int_{0}^{\pi} \int_{-\pi}^{\pi} \phi(u, v) \sin \mu u \cot \frac{u}{2} \sin \nu v \cot \frac{v}{2} \, du \, dv
\]

\[
= \int_{0}^{\pi} \cot \frac{u}{2} \sin \mu u du
\]

(3.12)

\[
\cdot \int_{-\pi}^{\pi} \cot \frac{v}{2} \left[ \frac{\partial}{\partial v} \int_{0}^{v} \phi(u, y) \sin \nu y \, dy \right] \, dv
\]

\[
= \int_{0}^{\pi} \cot \frac{u}{2} \sin \mu u \left\{ -\frac{\pi}{2} \cot \frac{1}{2n} \, c_{v} \left( u, \frac{1}{n} \right) 
\right. 
\]

\[
+ \frac{\pi}{4} \int_{-\pi}^{\pi} \csc^{2} \frac{v}{2} \, c_{v}(u, v) \, dv \right\} du,
\]

where $c_{v}(\alpha, \beta)$ denotes the $v$th Fourier coefficient of the odd function $\psi(u, \beta)$ which is equal to $\phi(u, v)$ for $0 \leq v \leq \beta$ and to zero for $\beta < v < \pi$. It follows from Young-Hausdorff's inequality that

\[
\left( \sum_{r=1}^{n} |c_{v}(u, v)|^k \right)^{1/k} \leq \left( \frac{1}{\pi} \int_{-\pi}^{\pi} |\psi(u, y)|^{k'} \, dy \right)^{1/k'}
\]

\[
= \left( \frac{1}{\pi} \int_{-\pi}^{\pi} |\phi(u, y)|^{k'} \, dy \right)^{1/k'},
\]

so that

\[
\left( \sum_{r=0}^{n} |I_{12}(\mu, \nu)|^k \right)^{1/k} \leq A \cot \frac{1}{2n} \int_{0}^{\pi} \mu \left( \int_{-\pi}^{\pi} |\phi(u, y)|^{k'} \, dy \right)^{1/k'} \, du
\]

\[
+ A \int_{-\pi}^{\mu} \mu du \int_{-\pi}^{\pi} \csc^{2} \frac{v}{2} \left( \int_{-\pi}^{v} |\phi(u, y)|^{k'} \, dy \right)^{1/k'},
\]

(3.13)

Hölder's inequality gives
\[
\int_0^{m^{-1}} \left( \int_{-v}^v |\phi(u, v)|^{k'} \, dy \right)^{1/k'} \, du \\
\leq m^{1/k'-1} \left( \int_0^{m^{-1}} \int_{-v}^v |\phi(u, v)|^{k'} \, dy \, du \right)^{1/k'} = m^{1/k'-1} \frac{v}{m} \left( \frac{v}{m} \right)^{1/k'}.
\]

Hence (3.13) is reduced to
\[
\left( \sum_{\mu=0}^n |I_{12}(\mu, \nu)|^k \right)^{1/k} \leq \mu \cot \frac{1}{2m} m^{1/k'-1} o(mn)^{-1/k'} + \mu m^{1/k'-1} \int_0^\pi \nu^{-2} \left( \frac{v}{m} \right)^{1/k'} \, dv = o(n)^{1/k}.
\]

Thus we obtain \( \left( \sum_{\mu=0}^m \sum_{\nu=0}^n |I_{12}(\mu, \nu)|^k \right)^{1/k} = o(mn)^{1/k} \). The integrals \( I_{22} \) and \( I_{32} \) may be treated in a similar manner as above. The following relations are thus established:

\[
(3.14) \quad \left( \sum_{\mu=0}^m \sum_{\nu=0}^n |I_{i2}(\mu, \nu)|^k \right)^{1/k} = o(mn)^{1/k} \quad (i = 1, 2, 3).
\]

Finally, we have to consider the integrals \( I_{13}, I_{23} \) and \( I_{33} \). The discussion of \( I_{13} \) is the same as \( I_{12} \), and the integral \( I_{23} \) has been treated implicitly in the discussion of \( I_{24} \). It remains therefore only to deal with \( I_{33} \). Regard the integrals

\[
I_{33}(\mu, \nu) = \int_{m^{-1}}^\pi \cos \mu \nu \, du \int_0^{n^{-1}} \phi(u, \nu) \sin \nu \cot \frac{\nu}{2} \, dv
\]

\( (\mu = 0, 1, 2, \cdots) \)
as the Fourier coefficients of the function of \( u \) which is equal to

\[
\int_0^{n^{-1}} \phi(u, \nu) \sin \nu \cot \frac{\nu}{2} \, dv \quad \text{for } m^{-1} \leq u \leq \pi,
\]

and to zero for \(-\pi \leq u < m^{-1} \), then by Hausdorff’s inequality,

\[
(3.15) \quad \left\{ \sum_{\mu=0}^m |I_{33}(\mu, \nu)|^k \right\}^{1/k} \leq \left( \frac{1}{4\pi^2} \int_{m^{-1}}^\pi \left| \int_0^{n^{-1}} \sin \nu \cot \frac{\nu}{2} \phi(u, \nu) \, dv \right|^{k'} \, du \right)^{1/k'}
\]

\[
\leq A_r \left( \int_{m^{-1}}^\pi \left( \int_0^{n^{-1}} |\phi(u, \nu)| \, dv \right)^{k'} \, du \right)^{1/k'}.
\]
It follows from Hölder's inequality that
\[ \int_0^1 \phi(u, v) \, dv \leq n^{1/k'-1} \left( \int_0^1 |\phi(u, v)|^{k'/k} \, dv \right)^{1/k'} \]
so that
\[ vn^{1/k'-1} \left( \int_{m-1}^x du \int_0^{n-1} |\phi(u, v)|^{k'/k} \, dv \right)^{1/k'} = vn^{1/k'-1}O(n^{-1/k'}) = O(1). \]

Hence from (3.15) it results that \( \sum_{\mu=0}^m \sum_{\nu=0}^n |I_{13}(\mu, \nu)|^{1/k} = o(mn)^{1/k}. \)

The following relations are thus proved:

\[ \left\{ \sum_{\mu=0}^m \sum_{\nu=0}^n |I_{13}(\mu, \nu)|^k \right\}^{1/k} = o(mn)^{1/k} \quad (i = 1, 2, 3). \]

Collecting the results (3.2), (3.3), (3.7), (3.11), (3.14), and (3.16) we obtain (3.1). Theorem I is thus proved.

4. Proof of Theorem II. On account of Theorem I, it suffices to show that the condition (1.3) is satisfied almost everywhere when \( s = f(x, y). \) Observing
\[ 4 |\phi_{x,y}(u, v)| \leq |f(x + u, y + v) - f(x, y)| \\
+ |f(x + u, y - v) - f(x, y)| \\
+ |f(x - u, y + v) - f(x, y)| \\
+ |f(x - u, y - v) - f(x, y)|, \]
and employing Minkowski's inequality, we immediately obtain the desired result from Lemma 2.

5. Proof of Theorem III. The proof depends upon the following two lemmas:

**Lemma 4.** Theorem III holds good when \( f(x, y) \) is bounded.

Since a bounded function belongs to \( L^p, p > 1, \) the lemma follows from Theorem II.

**Lemma 5.** Let \( h(x) \) be a function such that \( h \log^+ |h| \in L (-\pi, \pi). \) Let \( \beta_m = \beta_m(x, h) \) \((m = 0, 1, 2, \cdots)\) be the Fejér sums of the Fourier series of \( h(x), \) and \( \beta^*(x) = \sup_m |\beta_m(x)|, \) then
\[ \int_{-\pi}^{\pi} \beta^*(x) \, dx \leq A \int_{-\pi}^{\pi} |h| \log^+ |h| \, dx + B, \]
where \( A \) and \( B \) are absolute constants.
This lemma is due to Hardy and Littlewood [4]. See also [13, p. 248].

Before proving the theorem, we extend Lemma 5 to the case of two variables. Let, for fixed y,

\[ g(x, y) = \sup_\beta \beta_m(x; |f|). \]

Integrating this equation with respect to y, we obtain

\[ \int_{-\pi}^{\pi} \int_{-\pi}^{\pi} g(x, y) \, dx \, dy \]

\[ \leq 2\pi A \int_{-\pi}^{\pi} \int_{-\pi}^{\pi} |f(x, y)| \log^+ |f(x, y)| \, dx \, dy + 2\pi B. \]

Writing \( K_n(x) \) for the Fejér kernel, we have

\[ \sigma_{m,n}(x, y; f) = \frac{1}{\pi^2} \int_{-\pi}^{\pi} \int_{-\pi}^{\pi} f(u, v) K_m(x - u) K_n(y - v) \, dudv. \]

It follows that

\[ |\sigma_{m,n}(x, y; f)| \leq \frac{1}{\pi} \int_{-\pi}^{\pi} K_n(y - v) g(x, v) \, dv. \]

In virtue of Lebesgue’s theorem, the last expression tends to \( g(x, y) \) at almost every point \((x, y)\). Therefore the relation

\[ \sigma^*(x, y; f) = \limsup_{m,n \to \infty} |\sigma_{m,n}(x, y)| \leq g(x, y) \]

holds good almost everywhere. Combining this result with (5.1) we obtain

\[ \int_{-\pi}^{\pi} \int_{-\pi}^{\pi} \sigma^*(x, y; f) \, dx \, dy \]

\[ \leq 2\pi A \int_{-\pi}^{\pi} \int_{-\pi}^{\pi} |f(x, y)| \log^+ |f(x, y)| \, dx \, dy + 2\pi B. \]

Let \( \lambda \) be a positive constant. Substituting \( \lambda f \) for \( f \) in (5.2), we obtain

\[ \int_{-\pi}^{\pi} \int_{-\pi}^{\pi} \sigma^*(x, y; f) \, dx \, dy \]

\[ \leq 2\pi A \int_{-\pi}^{\pi} \int_{-\pi}^{\pi} |f(x, y)| \log^+ |f(x, y)| \, dx \, dy + 2\pi \frac{B}{\lambda}. \]
Let $\epsilon$ be a positive number; we take $\lambda$ so large that $2\pi B/\lambda < \epsilon/2$. Let
\[ f(x, y) = f'(x, y) + f''(x, y) \]
be such that $f'$ is bounded;
\begin{equation}
\int_{-\pi}^{\pi} \int_{-\pi}^{\pi} |f''(x, y)| \, dxdy < \epsilon, \tag{5.4}
\end{equation}
\begin{equation}
2\pi A \int_{-\pi}^{\pi} \int_{-\pi}^{\pi} |f''(x, y)| \log^+ |f''(x, y)| \, dxdy + 2\pi \frac{B}{\lambda}. \tag{5.5}
\end{equation}

Applying the inequality (5.3) to the function $f''(x, y)$, we obtain
\[ \int_{-\pi}^{\pi} \int_{-\pi}^{\pi} \sigma^*(x, y; f'') \, dxdy < \epsilon \]
by observing (5.5). Combining this relation with (5.4), we see that
the set $E(\epsilon)$ of points $(x, y)$ such that either $|f''(x, y)| > \epsilon^{1/2}$ or
$\sigma^*(x, y; f) > \epsilon^{1/2}$ is of plane measure less than $2\epsilon^{1/2}$. Now let $\sigma_{\mu, \nu}$ and $\sigma_{\mu, \nu}'$ denote respectively the $(\mu, \nu)$th Fejér sums of the Fourier series of $f'$ and $f''$, then
\[ \left( \sum_{\mu=0}^{m} \sum_{\nu=0}^{n} |\sigma_{\mu, \nu} - f|^k \right)^{1/k} \leq \left( \sum_{\mu=0}^{m} \sum_{\nu=0}^{n} |\sigma_{\mu, \nu}' - f|^k \right)^{1/k} + \left( \sum_{\mu=0}^{m} \sum_{\nu=0}^{n} |\sigma_{\mu, \nu}' - f''|^k \right)^{1/k}. \]

The first term on the right-hand side is $o(mn)^{1/k}$ almost everywhere, by Lemma 4. And
\[ \left( \sum_{\mu=0}^{m} \sum_{\nu=0}^{n} |\sigma_{\mu, \nu}' - f''|^k \right)^{1/k} \leq \left( \sum_{\mu=0}^{m} \sum_{\nu=0}^{n} |\sigma_{\mu, \nu}'|^k \right)^{1/k} + \left( \sum_{\mu=0}^{m} \sum_{\nu=0}^{n} |f''|^k \right)^{1/k} \leq [(m + 1)(n + 1)]^{1/k}(\sigma^*(x, y; f'') + |f'''|). \]

Hence, outside the set $E(\epsilon)$,
\[ \limsup_{m, n \to \infty} \left\{ \frac{1}{(m + 1)(n + 1)} \sum_{\mu=0}^{m} \sum_{\nu=0}^{n} |\sigma_{\mu, \nu} - f|^k \right\}^{1/k} \leq \sigma^*(x, y; f'') + |f'''| \leq 2\epsilon^{1/2}. \]

Since $\epsilon$ is arbitrary, the theorem follows.
BIBLIOGRAPHY


