HERMITIAN QUADRATIC FORMS IN A QUASI-FIELD

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1. Introduction. E. Witt proved the following theorem concerning quadratic forms in a fairly general field:

**Theorem 1.** Let $f_1 = ax_1^2 + \phi_1(x_2, \cdots, x_n)$ and $f_2 = ax_1^2 + \phi_2(x_2, \cdots, x_n)$ be quadratic forms whose coefficients lie in a given field $F$ in which $2 \neq 0$. Then the equivalence in $F$ of $f_1$ and $f_2$ implies that of $\phi_1$ and $\phi_2$.

It is our purpose here to generalize this theorem to any quasi-field (a field, except that multiplication may not be commutative) on which is defined a conjugate operation of period 2 with the usual properties

$$\overline{a + b} = \overline{a} + \overline{b}, \quad \overline{ab} = \overline{b} \cdot \overline{a}.$$ 

Well known examples are any field with $\overline{a} = a$; the field of complex numbers with the usual complex conjugate; the system of quaternions with real coefficients and the usual conjugate. The analogue in a quasi-field of quadratic form in a field is the hermitian quadratic form

$$f = \overline{x}'Ax = \sum_{i,j=2}^n \overline{x}_i a_{ij} x_j,$$

where $A' = A$, or $a_{ij} = a_{ji}$.

The scalars of a quasi-field are the elements $s$ such that $\overline{s} = s$. The diagonal elements of a hermitian matrix are therefore scalars. The process of completing squares is carried out in much the same way as in a field. Thus if, in $f$ above, $a_{11} \neq 0$,

$$f = (\overline{x}_1 + \sum_{i=1}^n \overline{x}_i a_{i1} a_{11}^{-1}) a_{11} (x_1 + \sum_{i=2}^n a_{11}^{-1} a_{1i} \overline{x}_i) + \sum_{j,k=2}^n \overline{x}_j (a_{jk} - a_{kj} a_{11}^{-1} a_{1k}) \overline{x}_k.$$

Hence the analogue of a form like $f_1$ in Witt's theorem can be written

$$\overline{x}_1 a x_1 + \phi, \quad \text{where} \quad \phi = \sum_{i,j=2}^n \overline{x}_i b_{ij} x_j, \quad \overline{b}_{ij} = b_{ji}.$$

Since determinants do not exist in a quasi-field (except for hermitian matrices), we cannot demonstrate that a matrix $T$ is nonsingular.
by the nonvanishing of a determinant. Instead, we may construct explicitly the reciprocal matrix $V$, such that $VT = TV = I$.

We shall, in §3, consider automorphs of $f$, that is matrices $T$ such that $T^\prime AT = A$. If $z$ denotes the first column of $T$, then $z^\prime Az = a_{11}$, that is $z$ is a representation of the leading coefficient $a_{11}$ of $f$. If $a_{11} \neq 0$, we shall for any given representation $z$ of $a_{11}$ construct a corresponding automorph of $f$.

2. A generalization of Witt's theorem. The theorem we shall prove is the following.

**Theorem 2.** Let $F$ be a quasi-field with a conjugate operation as described above, $2 \neq 0$. Let $a$ be a nonzero scalar, and $B_1, B_2$ nonsingular hermitian matrices of order $n - 1$, with elements in $F$. Let

$$A_1 = \begin{bmatrix} a & 0' \\ 0 & B_1 \end{bmatrix}, \quad A_2 = \begin{bmatrix} a & 0' \\ 0 & B_2 \end{bmatrix},$$

where $0$ denotes a column, and $0'$ a row, of $n - 1$ zeros. Let $T$ denote any transformation (with coefficients in $F$) of $A_1$ into $A_2$, that is let

$$A_2 = T^\prime A_1 T.$$

Then we can construct a transformation of $B_1$ into $B_2$.

**Proof.** We can write (2) in the form

$$A_2 = \begin{bmatrix} a & 0' \\ 0 & B_2 \end{bmatrix} = \begin{bmatrix} x_0 & \hat{y} \\ \hat{x} & T_1 \end{bmatrix} \begin{bmatrix} a & 0' \\ 0 & B_1 \end{bmatrix} \begin{bmatrix} x_0 & y' \\ \hat{x}' & \hat{T}_1' \end{bmatrix},$$

where $x_0$ is a constant, $x$ and $y$ are column vectors of $n - 1$ components, $T_1$ a matrix of order $n - 1$. Expanding (3) we get

$$x_0 ay' + \hat{x}' B_1 x = 0,$$

$$y_0 ax_0 + \hat{T}_1' B_1 x = 0,$$

$$y_0 ay' + \hat{T}_1' B_1 x = 0.$$

Our problem is to derive from (4)–(6) a transformation of $B_1$ into $B_2$.

Suppose we could secure $x_0 = 1, x = 0$, to begin with. Then by (5), $y' = 0', \hat{y} = 0$; and by (6), $T_1' B_1 T_1 = B_2$. What we shall do is construct a nonsingular automorph $U$ of $A_1$ whose first column is the same as that of $T$. Having done this, let

$$W = \begin{bmatrix} x_0 & w' \\ z & U_1 \end{bmatrix}$$

be the reciprocal of $U = \begin{bmatrix} x_0 & \cdot \\ x & \cdot \end{bmatrix}$.

Then $x_0 x_0 + w' x = 1, z x_0 + U_1 x = 0$, and so
say; and \( WT \) also replaces \( A_1 \) by \( A_2 \). By the preceding remark, \( V'B_1V = B_2 \).

Hence Theorem 2 is a consequence of the following theorem; or see §4.

3. Automorphs with an assigned first column. We now prove:

**Theorem 3.** Let \( A = (a_{ij}) \) be any nonsingular hermitian matrix with coefficients in a quasi-field \( F \) of characteristic not 2. Let \( z \) be a representation in \( F \) of \( a_{11} \), that is \( z'Az = a_{11} \), and assume \( a_{11} \neq 0 \). Then there exists in \( F \) a nonsingular automorph of \( A \) with \( z \) as its first column.

We first complete squares, which amounts to applying a transformation

\[
WT = \begin{bmatrix} z_0 & w' \\ z & U_1 \end{bmatrix} \begin{bmatrix} x_0 & y' \\ x & T_1 \end{bmatrix} = \begin{bmatrix} 1 & u' \\ 0 & V \end{bmatrix},
\]

Here \( a = a_{11} \), and \( B_1 \) is hermitian with \( A \). We have \( z'Az = u'A_1u \), where \( z = Pu \); and if we can construct an automorph \( U \) of \( A_1 \) with \( u \) as first column, then \( PUP^{-1} \) will be an automorph of \( A \) with \( z \) as first column. For, the first column of \( PU \) is \( Pu \), and multiplication on the right by \( P^{-1} \) does not change the first column of \( PU \).

We can therefore use the notations (4)–(6) with \( B_2 = B_1 \). Here \( x_0 \) and \( x \) are given as satisfying (4), and it is required to find \( y \) and \( T_1 \) to satisfy (5) and (6).

The cases \( x_0 = 0 \) and \( x_0 \neq 0 \) must be distinguished.

Let \( x_0 = 0 \). Then \( x'B_1x = a \), and we must choose \( y \) and \( T_1 \) to satisfy

\[
(8) \quad x'B_1T_1 = 0', \quad yay' + T_1B_1T_1 = B_1.
\]

The last equation can be replaced by \( (xy')'B_1(xy') + T_1'B_1T_1 = B_1 \), hence by

\[
(9) \quad (xy' + T_1)'B_1(xy' + T_1) = B_1,
\]

in view of (8). Then all of (8) will hold if we put

\[
(10) \quad T_1 = I - xy', \quad x'B_1 - ay' = 0;
\]

the last is satisfied if \( y' = a^{-1}x'B_1 \). It will be found that \( y'x = 1 \), and

\[
(11) \quad \begin{bmatrix} 0 & y' \\ x & T_1 \end{bmatrix} \begin{bmatrix} 0 & y' \\ x & T_1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & I \end{bmatrix}.
\]
Let \( x_0 \neq 0 \). Then (5) will hold if

\[
(12) \quad y = -T_1' B_1 x x_0^{-1} a^{-1}, \quad ay' = -x_0^{-1} x' B_1 T_1,
\]

and then (6) becomes

\[
(13) \quad T_1'(B_1 + B_1 x(x_0 a x_0)^{-1} x'B_1) T_1 = B_1.
\]

We therefore try to choose a number \( k \) in \( F \) to satisfy

\[
(14) \quad (I + B_1 x k x') B_1(I + x k x'B_1) = B_1 + B_1 x(x_0 a x_0)^{-1} x'B_1.
\]

In virtue of (4) this will be satisfied if

\[
(15) \quad k + k + k(a - x_0 a x_0)k = (x_0 a x_0)^{-1}.
\]

Here we try the substitution \( k = t^{-1} \), and multiply left and right by \( t \) and \( i \) to obtain

\[
(16) \quad i + t + a - x_0 a x_0 = t(x_0 a x_0)^{-1} i.
\]

To satisfy this we put \( t = k + x_0 a x_0 \), and find

\[
(17) \quad h(x_0 a x_0)^{-1} h = a, \text{ which holds if } h = \pm a x_0, \quad t = (x_0 \pm 1) a x_0.
\]

Since \( 2 \neq 0 \) in \( F \) we can choose the sign to make \( t \neq 0 \), so that \( k \) exists.

We can now solve for \( T_1 \) the equations

\[
(18) \quad (I + x k x'B_1) T_1 = I = T_1(I + x k x'B_1).
\]

For if we put \( T_1 = I + x m \tilde{x}' B_1 \), where \( m \) is a constant to be determined, then (18) will hold if

\[
(19) \quad k + m + k(a - x_0 a x_0)m = 0 = m + k + m(a - x_0 a x_0)k.
\]

Noting that \( k i = i k = 1 \) (since \( k = t^{-1} \)), we replace (19) by

\[
(20) \quad m^{-1} + i + a - x_0 a x_0 = 0 = i + m^{-1} + a - x_0 a x_0,
\]

which holds if \( m^{-1} = -i - a + x_0 a x_0 = -(1 \pm x_0) a \). Thus \( m \) exists, and

\[
(21) \quad T_1 = I + x m \tilde{x}' B_1.
\]

Finally we verify that the automorph so constructed is nonsingular. By (3),

\[
(22) \quad \begin{bmatrix}
    a^{-1} \tilde{x}_0 a & a^{-1} \tilde{x}' B_1 \\
    B_1^{-1} \tilde{y} a & B_1^{-1} T_1' B_1
\end{bmatrix}
\begin{bmatrix}
    x_0 & y' \\
    x & T_1
\end{bmatrix}
= \begin{bmatrix}
    1 & 0 \\
    0 & I
\end{bmatrix}.
\]

Hence we have only to prove that

\[8 The referee remarked that (23) need not be proved since it is known that if \((a)(\beta) = 1\), where \((a)\), \((\beta)\) are matrices with elements in a division ring, then also \((\beta)(a) = 1\).
Put \( r = \bar{x}_0 ax_0 \). Using \( \bar{x}'B_1x = a - r \), and verifying that \( m + \bar{m} + m(a - r)\bar{m} = -a^{-1} \), the proof of (23) is easy. For example, \( x_0 a^{-1}\bar{x}_0 a + y'\bar{B}_1^{-1}y a = 1 \) if and only if \( ra^{-1}r + \bar{x}'B_1(I + x_m\bar{x}'B_1)\bar{B}_1^{-1}B_1^{-1}(I + B_1x\bar{m}\bar{x}')B_1x = r \), or \( ra^{-1}r + a - r + (a - r)(-a^{-1})(a - r) = r \), or \( r = r \).

One additional remark is worth making for the case where \( F \) is a (commutative) field. If \( K \) is any square matrix of rank 1, it can of course be expressed as \( xy' \), where \( x \) and \( y \) are column vectors. The determinant of \( I + K \) is readily found, since as is easily seen,

\[
|I + xy'| = 1 + x'y = 1 + y'x = 1 + \sum x_iy_i.
\]

The reciprocal of a matrix of the type \( I + hxy' \) (where \( h \) is a constant) can be found by noting that

\[
(I + hxy')(I + kxy') = I + \{h + k + hky'x\}xy',
\]

and choosing \( k \) to make \( h + k + hky'x = 0 \).

4. An alternative construction for Theorem 2. For some purposes, it is more advantageous not to construct an automorph of \( A_1 \), but to continue the argument from (4)–(6) as follows. If \( x_0 = 0 \), \( xy' + T_1 \) replaces \( B_1 \) by \( B_2 \). Let \( x_0 \neq 0 \). Then (5) is equivalent to (12), and (6) reduces to (13) with \( B_2 \) instead of \( B_1 \) on the right. We have (14)–(17) as before, and so \( T_1 + x\bar{k}\bar{x}'B_1T_1 \) is a transformation replacing \( B_1 \) by \( B_2 \).

This transformation may in certain cases be integral (in a sense which we need not discuss here) even though no integral automorph of \( A_1 \) exists with \( x_0 \) and \( x \) as first column.

It should be mentioned that the preceding methods can be extended to the case where the element \( a \) is replaced by a nonsingular hermitian matrix of order greater than 1.

BIBLIOGRAPHY


McGill University