Recently Littlewood and Offord\textsuperscript{1} proved the following lemma: Let $x_1, x_2, \ldots, x_n$ be complex numbers with $|x_i| \geq 1$. Consider the sums $\sum_{k=1}^{n} \epsilon_k x_k$, where the $\epsilon_k$ are $\pm 1$. Then the number of the sums $\sum_{k=1}^{n} \epsilon_k x_k$ which fall into a circle of radius $r$ is not greater than

$$cr2^n(n \log n)^{-1/2}.$$ 

In the present paper we are going to improve this to

$$cr2^n n^{-1/2}.$$ 

The case $x_i = 1$ shows that the result is best possible as far as the order is concerned.

First we prove the following theorem.

\textbf{Theorem 1.} Let $x_1, x_2, \ldots, x_n$ be $n$ real numbers, $|x_i| \geq 1$. Then the number of sums $\sum_{k=1}^{n} \epsilon_k x_k$ which fall in the interior of an arbitrary interval $I$ of length 2 does not exceed $C_{n,m}$, where $m = \lfloor n/2 \rfloor$. ($\lfloor x \rfloor$ denotes the integral part of $x$.)

Remark. Choose $x_i = 1$, $n$ even. Then the interval $(-1, +1)$ contains $C_{n,m}$ sums $\sum_{k=1}^{n} \epsilon_k x_k$, which shows that our theorem is best possible.

We clearly can assume that all the $x_i$ are not less than 1. To every sum $\sum_{k=1}^{n} \epsilon_k x_k$ we associate a subset of the integers from 1 to $n$ as follows: $k$ belongs to the subset if and only if $\epsilon_k = +1$. If two sums $\sum_{k=1}^{n} \epsilon_k x_k$ and $\sum_{k=1}^{n} \epsilon'_k x_k$ are both in $I$, neither of the corresponding subsets can contain the other, for otherwise their difference would clearly be not less than 2. Now a theorem of Sperner\textsuperscript{2} states that in any collection of subsets of $n$ elements such that of every pair of subsets neither contains the other, the number of sets is not greater than $C_{n,m}$, and this completes the proof.

An analogous theorem probably holds if the $x_i$ are complex numbers, or perhaps even vectors in Hilbert space (possibly even in a Banach space). Thus we can formulate the following conjecture.

\textbf{Conjecture.} Let $x_1, x_2, \ldots, x_n$ be $n$ vectors in Hilbert space, $\|x_i\| \geq 1$. Then the number of sums $\sum_{k=1}^{n} \epsilon_k x_k$ which fall in the interior of an arbitrary sphere of radius 1 does not exceed $C_{n,m}$. 

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At present we cannot prove this, in fact we cannot even prove that the number of sums falling in the interior of any sphere of radius 1 is \( o(2^n) \).

From Theorem 1 we immediately obtain the following corollary.

**Corollary.** Let \( r \) be any integer. Then the number of sums \( \sum_{k=1}^{n} e_kx \) which fall in the interior of any interval of length \( 2r \) is less than \( rC_{n,m} \).

**Theorem 2.** Let the \( x_i \) be complex numbers, \( |x_i| \geq 1 \). Then the number of sums \( \sum_{k=1}^{n} e_kx \) which fall in the interior of an arbitrary circle of radius \( r \) (\( r \) integer) is less than

\[
\sigma rC_{n,m} < c_2r^{2^n/n^{1/2}}.
\]

Thus the total number of sums which fall in the interior of a circle of radius \( r \) is less than

\[
c_2r^{2^n/n^{1/2}},
\]

which completes the proof.

Our corollary to Theorem 1 is not best possible. We prove:

**Theorem 3.** Let \( r \) be any integer, the \( x_i \) real, \( |x_i| \geq 1 \). Then the number of sums \( \sum_{k=1}^{n} e_kx \) which fall into the interior of any interval of length \( 2r \) is not greater than the sum of the \( r \) greatest binomial coefficients (belonging to \( n \)).

Clearly by choosing \( x_i = 1 \) we see that this theorem is best possible. The same argument as used in Theorem 1 shows that Theorem 3 will be an immediate consequence of the following theorem.

**Theorem 4.** Let \( A_1, A_2, \ldots, A_u \) be subsets of \( n \) elements such that no two subsets \( A_i \) and \( A_j \) satisfy \( A_i \supseteq A_j \) and \( A_i \cap A_j \) contains more than \( r - 1 \) elements. Then \( u \) is not greater than the sum of the \( r \) largest binomial coefficients.

Let us assume for sake of simplicity that \( n = 2m \) is even and \( r = 2j + 1 \) is odd. Then we have to prove that
Our proof will be very similar to that of Sperner. Let $A_1, A_2, \ldots, A_u$ be a set of subsets which have the required property and for which $u$ is maximal. It will suffice to show that in every $A$ the number of elements is between $n-j$ and $n+j$. Suppose this were not so, then by replacing if need be each $A$ by its complement we can assume that there exist $A$'s having less than $n-j$ elements. Consider the $A$'s with fewest elements; let the number of their elements be $n-j-y$ and let there be $x$ $A$'s with this property. Denote these $A$'s by $A_1, A_2, \ldots, A_x$. To each $A_i, i=1, 2, \ldots, x$, add in all possible ways $r$ elements from the $n+j+y$ elements not contained in $A$. We clearly can do this in $C_{n+j+y, r}$ ways. Thus we obtain the sets $B_1, B_2, \ldots$, each having $n+j-y+1$ elements. Clearly each set can occur at most $C_{n+j+y, r}$ times among the $B$'s. Thus the number of different $B$'s is not less than

$$xC_{n+j+y, r}(C_{n+j+y, r})^{-1} > x.$$
of Menger.\footnote{See, for example, D. König, *Theorie der endlichen und unendlichen Graphen*, p. 244.} Let $G$ be any graph, $V_1$ and $V_2$ two disjoint sets of its vertices. Assume that the minimum number of points needed for the separation of $V_1$ and $V_2$ is $w$. Then there exist $w$ disjoint paths connecting $V_1$ and $V_2$. (A set of points $w$ is said to separate $V_1$ and $V_2$, if any path connecting $V_1$ with $V_2$ passes through a point of $w$.)

Hence the proof of our lemma will be completed if we can show that the vertices $V_1$ containing $n - j - y$ elements can not be separated from the vertices $V_2$ containing $n + j + y$ elements by less than $C_{2n, n-j-y}$ vertices. A simple computation shows that $V_1$ and $V_2$ are connected by

$$C_{2n, n-j-y}(n + j + y)(n + j + y - 1) \cdots (n - j - y + 1)$$

paths. Let $z$ be any vertex containing $n + i$ elements, $-j - y \leq i \leq j + y$. A simple calculation shows the the number of paths connecting $V_1$ and $V_2$ which go through $z$ equals

$$(n+i)(n+i-1) \cdots (n-j-y+1)(n-i)(n-i-1) \cdots (n-j-y+1)$$

$$\leq (n+j+y)(n+j+y-1) \cdots (n-j-y+1).$$

Thus we immediately obtain that $V_1$ and $V_2$ can not be separated by less than $C_{2n, n-j-y}$ vertices, and this completes the proof of our lemma.

Let now $A_1, A_2, \ldots, A_x$ be the $A$'s containing $n - j - y$ elements. By our lemma there exist sets $A_i, i = 1, 2, \ldots, x; l = 1, 2, \ldots, 2j + 2y + 1$, such that $A_i(2j + 2y + 1)$ has $n + j + y$ elements and $A_i \subset A_{i+1}$ and all the $A$'s are different. Clearly not all the sets $A_i$, $l = 1, 2, \ldots, 2j + 2y + 1$, can occur among the $A_1, A_2, \ldots, A_x$. Let $A_1$ be the first $A$ which does not occur there. Evidently $s \leq r$. Omit $A_1$ and replace it by $A_1(2j + 2y + 1)$. Then we get a new system of sets having also $u$ elements which clearly satisfies our conditions, and where the sets containing fewest elements have more than $n - j - y$ elements and the sets containing most elements have not more than $n + j + y$ elements. By repeating the same process we eventually get a system of $A$'s for which the number of elements is between $n - j$ and $n + j$. This shows that

$$u \leq \sum_{i=m-j}^{m+j} C_{2n, n+i},$$

which completes the proof.

One more remark about our conjecture: Perhaps it would be easier to prove it in the following stronger form: Let $|x_i| \geq 1$, then the num-
The number of sums $\sum_{k=1}^{n} \varepsilon_k x_k$ which fall in the interior of a circle of radius 1 plus one half the number of sums falling on the circumference of the circle is not greater than $C_{n,m}$. If the $x_i$ are real it is quite easy to prove this.

We state one more conjecture.

(1). Let $|x_i| = 1$. Then the number of sums $\sum_{k=1}^{n} \varepsilon_k x_k$ with $|\sum_{k=1}^{n} \varepsilon_k x_k| \leq 1$ is greater than $c2^{n-1}$, $c$ an absolute constant.