NULL SYSTEMS IN PROJECTIVE SPACE

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If $P$ is an (abstract) $n$-dimensional projective space, then we define a polarity in $P$ as a correspondence $\phi$ associating with every point $Q$ in $P$ a hyperplane $Q^\phi$ and with every hyperplane $h$ in $P$ a point $h^\phi$ in such a way that:

(i) $Q = Q^\phi$ for every point $Q$ and $h = h^\phi$ for every hyperplane $h$.

(ii) The point $Q$ is on the hyperplane $h$ if, and only if, the hyperplane $Q^\phi$ passes through the point $h^\phi$.

It is an immediate consequence of (i) that polarities are 1:1 correspondences.

We shall term $\phi$ a null-polarity if the polarity $\phi$ has the additional property that:

(iii) Every point $Q$ is on the corresponding hyperplane $Q^\phi$, and consequently every hyperplane $h$ passes through the corresponding point $h^\phi$.

Extending a result of Veblen and Young, R. Brauer\(^1\) has shown that the existence of a null-polarity in $P$ implies that the number $n$ of dimensions of $P$ is odd, and he has connected the null-polarities with the so-called null-systems, provided $P$ is the $n$-dimensional projective space over a commutative field of coordinates. It is the object of the present note to show that this last hypothesis may be omitted; more precisely we are going to show that if the dimension of $P$ is greater than 1, then the existence of a null polarity is equivalent to the fact that $P$ is of odd dimension and is a projective space over a commutative field of coordinates.

If $P$ is a projective space of dimension 1, then the hyperplanes are points too. The identity transformation on the points of the line $P$ is therefore the null-polarity of $P$. For this reason we shall assume throughout the remainder of this note that $P$ be of dimension greater than 1.

The case of a projective plane $P$ has to be treated separately from the others, since the Theorem of Desargues need not hold true in a projective plane, though it is true for all the higher-dimensional projective spaces.

A projective plane is a system of points and lines such that any two different lines meet in one and only one point, any two different

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Lemma. If $p$ is a polarity of the projective plane $P$, if $Q$ and $R$ are different points in $P$ such that $Q$ is on the line $Q^p$ and $R$ is on the line $R^p$, then $Q$ is not on $R^p$ (nor is $R$ on $Q^p$).

Proof. If $Q$ were on $R^p$, then $R^p$ would be the uniquely determined line through the two different points $R$ and $Q$, since $R$ is on $R^p$. Furthermore $R$ would be on $Q^p$; and it would follow likewise that $Q$ is the uniquely determined line through $R$ and $Q$. Thus $R^p = Q^p$ and hence $R = Q$, a contradiction proving our contention.

Corollary. There does not exist a null-polarity in a projective plane.

This is an immediate consequence of the lemma.

Because of the corollary we shall assume throughout the remainder of this note that the dimension of the projective space $P$ is at least 3. In this case $P$ is the $n$-dimensional projective space over an essentially uniquely determined, not necessarily commutative, field $F$.

If $F$ is any field (commutative or not), and if $n$ is an integer not less than 3, then we denote by $(F, n)$ an additively written abelian group, admitting the elements in $F$ as left-multipliers, and having the rank $n + 1$ over $F$. The $n$-dimensional projective space over $F$ is then essentially the same as the partially ordered set $P(F, n)$ of all the $F$-admissible subgroups of $(F, n)$, the points being of the form $Fx$ with $x \neq 0$, and the hyperplanes being of rank $n$.

Every polarity $p$ of $P(F, n)$ may, as is well known, be represented in the following form: There exist an anti-automorphism $a$ of $F$ (satisfying $a^2 = 1$) and an $F$-valued function $f(x, y)$, for $x, y$ in $(F, n)$, satisfying:

\begin{align*}
f(x, y) &= 0 \text{ if, and only if, } f(y, x) = 0; \\
f(ux + vy, z) &= uf(x, z) + vf(y, z), \quad f(z, ux + vy) = f(z, x)u^a + f(z, y)v^a.
\end{align*}

The point $Fx$ is on the hyperplane $(Fy)^p$ if, and only if, $f(x, y) = 0$.

Assume now that the polarity $p$ be a null-polarity. This is equivalent to saying $f(x, x) = 0$ for every $x$ in $(F, n)$. If $x \neq 0$, then there exists

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$^2$ A proof of this fact may be effected in essentially the same fashion as done by R. Brauer, op. cit. pp. 251, 252, in the case of commutative $F$; for a detailed proof of a more comprehensive fact see R. Baer, A unified theory of projective spaces and finite abelian groups, Trans. Amer. Math. Soc. vol. 52 (1942) pp. 315–317.

$^3$ We state here only such properties of the function $f(x, y)$ as will be needed later. Further properties have to be imposed to assure that, conversely, every such $f(x, y)$ defines a polarity. Note in particular that no use has been made of the involutorial character of $a$. 

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at least one \( y \) such that \( f(x, y) \neq 0 \), since otherwise \( (Fx)^p \) would be the whole space instead of only a hyperplane. Let \( x, y \) be any pair of elements such that \( f(x, y) \neq 0 \). If \( t \) is some element in \( F \), then we find

\[
0 = f(x + ty, x + ty) = f(x, x) + tf(y, x) + tf(y, y) = f(x, y)^a + tf(y, x).
\]

Substituting \( t = 1 \), we obtain \( f(x, y) + f(y, x) = 0 \); and thus the above equation reduces to \( 0 = f(x, y)^a - tf(x, y) \). Since \( f(x, y) \neq 0 \), this implies \( t^a = f(x, y)^{-1}tf(x, y) \) for every \( t \) in \( F \), proving that the anti-automorphism \( a \) of \( F \) is an inner automorphism of \( F \). Hence \( F \) is commutative and \( a = 1 \). Combining this with the fact that \( f(x, y) = 0 \) if, and only if, \( f(y, x) = 0 \), we see that \( f(x, y) \) is actually a skew-symmetric bilinear form.

It is well known\(^4\) that there exists to every skew-symmetric, \( F \)-valued bilinear form \( f(x, y) \) over \( (F, n) \) a basis \( x(1), \ldots, x(m), y(1), \ldots, y(m), z(1), \ldots, z(k) \) of \( (F, n) \), meeting the following requirements:

(a) \( 0 \leq m, \quad 0 \leq k, \quad 2m + k = n + 1 \);

(b) \( f(x(i), y(i)) = 1 \) for \( 1 \leq i \leq m \);
\[
f(x(i), x(j)) = f(y(i), y(j)) = 0 \quad \text{for every } i \text{ and } j;
\]

(c) \( f(x(i), z(j)) = f(y(i), z(j)) = f(z(i), z(j)) = 0 \) for every \( i \) and \( j \);
\[
f(x(i), y(j)) = 0 \quad \text{for every } i \neq j.
\]

This implies in particular \( f(x, z(i)) = 0 \) for every \( x \) in \( (F, n) \) so that the hyperplane \( (Fx(i))^p \) would contain every point of the space, an impossibility proving \( k = 0 \) and \( n = 2m - 1 \). Summarizing our results we obtain:

If \( P \) is a projective space of dimension \( n \), greater than 1, and if \( p \) is a null-polarity in \( P \), then \( n = 2m - 1 \) for \( m \) a positive integer, \( P \) is the \( n \)-dimensional projective space over a commutative field \( F \), and there exists a system of homogeneous coordinates in \( P \) such that the point \( F(x_0, \ldots, x_{2m-1}) \) is on the hyperplane \( [F(y_0, \ldots, y_{2m-1})]^p \) if, and only if,

\[
0 = \sum_{i=0}^{m-1} (x_{2i+1}y_{2i} - y_{2i+1}x_{2i}).
\]

Combining all our results one deduces without difficulty the following facts.

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Theorem A. In an n-dimensional projective space there exists essentially at most one null-polarity.

Theorem B. In the n-dimensional projective space $P$ with $1 < n$ there exists a null-polarity if, and only if, $n$ is odd and $P$ is the projective space of dimension $n$ over a commutative field.

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