A NOTE ON THE FIRST NORMAL SPACE OF
A $V_m$ IN AN $R_n$

YUNG-CHOW WONG

Let $N$ be the normal plane at a point $p$ of a surface $V_2$ in a Euclidean 4-space $R_4$. Calapso proved that the hypersphere $S$ in $R_4$ passing through $p$ and with center $c$ in $N$ cuts $V_2$ in a curve with a double point at $p$, at which the two tangents to the curve coincide if and only if $c$ lies on the Kommerell conic. The Kommerell conic is the locus of the point in which $N$ (at $p$) is cut by the neighboring normal planes of $V_2$.

The purpose of this note is to generalize this result to the case of a subspace $V_m$ in a Euclidean $n$-space $R_n$. To do this we shall first state some definitions and known results concerning the first (or principal) normal space of $V_m$ in $R_n$.

Let $X^k (k = 1, \cdots, n)$ be the rectangular cartesian coordinates in $R_n$ and let

(1) \[ X^k = x^k(u^a) \] \[ (a, b, c = 1, \cdots, m) \]

be the equations of a $V_m$. Put

(2) \[ B_{a}^k = \partial_{a}x^k \equiv \partial x^k / \partial u^a. \]

Then the fundamental tensor and curvature tensor of $V_m$ in $R_n$ are, respectively,

(3) \[ 'g_{cb} = \sum_k B_{c}^k B_{b}^k, \]

(4) \[ H_{cb}^k = \partial_c B_{b}^k - 'T_{cb}^a B_{a}^k, \]

where $'T_{cb}^a$ is the Christoffel symbol of the second kind for $V_m$.

Let us consider the figure surrounding a certain point $p$ of $V_m$. We have at $p$ a tangent $m$-plane and a normal $(n-m)$-plane $N$. Let $i^a$ be the unit tangent vector at $p$ of an arbitrary curve in $V_m$ passing through $p$. Then the component in $N$ of the first curvature vector of the curve with respect to $R_n$ is
The vector \( u^k \) spans the first normal \( m' \)-plane \( N' \) (in \( N \)) of \( V_m \) in \( R_n \).

The arithmetic mean of the vector \( u^k \) for \( m \) mutually orthogonal normal curvature vector

\[
M^k = \frac{1}{m} \sum_{i=1}^{m} u^i.
\]

Any vector \( n_k (= w^k) \) in \( N \) orthogonal to \( N' \) is such that

\[
n_k H'_{cb} = 0.
\]

\( p \) is called a semi-umbilical point if there exists a vector \( v_h \) such that

\[
v_k H'_{cb} = g_{cb}
\]

is satisfied. Because of (7) we may suppose that \( v_k \) is a vector in \( N' \).

The normal \((n-m)\)-plane at the neighboring point \( p + dp \) may or may not intersect \( N' \) (at \( p \)) at points other than \( p \) depending on the direction of \( dp \). But we call the locus of the intersection of \( N' \) at \( p \) (the point \( p \) being excluded) by the normal \((n-m)\)-planes of all the neighboring points \( p + dp \) the K-variety at \( p \) of \( V_m \) in \( R_n \). The equation of the K-variety is

\[
\text{Det} (Y_k H'_{cb} - g_{cb}) = 0,
\]

where \( Y_k \) is a variable vector in \( N' \). The K-variety is an algebraic hypersurface of order \( m \) in \( N' \). At a semi-umbilical point, it is a hypercone in \( N' \) with vertex at the end point \( v(x^k + v^k) \) (cf. (8)).

We are now in a position to prove the following theorems.

**Theorem 1.** The hypersphere \( S \) in \( R_n \) passing through \( p \) and with center at a point \( c \) in \( N \) intersects \( V_m \) in a \( V_m^{-1} \) with a singular point at \( p \). The tangent lines to \( V_m^{-1} \) at \( p \) form a hypercone \( C \) (in the tangent \( m \)-plane to \( V_m \)) of generally the second degree.

**Theorem 2.** \( p \) is semi-umbilical if and only if there exists a hypercone \( C \) at \( p \) which is of at least the third degree.

**Theorem 3.** All the points in \( N \) with the same projection in \( N' \) have the same hypercone \( C \). There exist two points in \( N' \) having the same hypercone \( C \) if and only if \( p \) is semi-umbilical. At a semi-umbilical point all the points (with the exception of the point \( v (x^k + v^k) \)) on each line in \( N' \) passing through \( v \) have the same hypercone \( C \). No two points in \( N' \) noncollinear with \( v \) have the same hypercone \( C \).

**Theorem 4.** The locus of the point in \( N' \) whose hypercone \( C \) sustains
an orthogonal enuple of generators is the polar hyperplane of the end point of the mean normal curvature vector with respect to the unit hyper-sphere in \( N' \) (with center at \( p \)).

**Theorem 5.** The \( K \)-variety is the locus of the point in \( N' \) whose hypercone \( C \) has a line of vertices.

Theorem 5 for \( m = 2, n = 4, N' = N \) reduces to the theorem of Calapso quoted at the beginning of this paper.

**Proof.** The expansion of \( x^k(u^a) \) in the neighborhood of \( p : x^k_0 = x^k(u^a_0) \) is

\[
x^k = x^k_0 + (\partial_a x^k_0) du^a + 2^{-1}(\partial_a \partial_b x^k_0) du^a du^b + \cdots.
\]

But by (2) and (4),

\[
\partial_a x^k = B^k_a, \quad \partial_c \partial_b x^k = \partial_c B^k_b = H_{cb}^k + 'T_{cb}^a B^k_a.
\]

Therefore

\[
(10) \quad x^k = x^k_0 + (B^k_0) du^a + 2^{-1}(H_{cb}^k + 'T_{cb}^a B^k_a) du^a du^b + \cdots.
\]

The equation of the hypersphere \( S \) in \( R^m \) passing through \( p \) and with center at a point \( c(x^k_0 + c^k) \) in \( N \) is

\[
\sum_k \left( X^k - x^k_0 - c^k \right)^2 = \sum_k \left( c^k \right)^2.
\]

Using (1) for \( X^k \) we see that \( S \) intersects \( V_m \) at the points \( (u^a_0 + du^a) \) at which

\[
\sum_k \left[ -c^k + (B^k_0) du^a + 2^{-1}(H_{cb}^k + 'T_{cb}^a B^k_a) du^a du^b + \cdots \right]^2 = \sum_k \left( c^k \right)^2.
\]

(11)

Let us arrange this in powers of \( du^a \). Then the constant term disappears. The first term vanishes because \( c^k \) is orthogonal to the tangent \( m \)-plane spanned by \( (B^k_0) \):

\[
\sum_k c^k (B^k_0) = 0.
\]

(12)

The second degree term is

\[
\left[ \sum_k (B^k_b B^k_c)_0 - \sum_k c^k (H_{cb}^k + 'T_{cb}^a B^k_a)_0 \right] du^a du^b
\]

\[
= \left( g_{cb} - c_b H_{cb}^k \right) du^a du^b.
\]
by (3) and (12). This proves Theorem 1 and gives us the equation of the hypercone $\mathcal{C}$ as

$$\langle c_k H^{\cdot \cdot k}_{eb} - 'g_{eb} \rangle Z^c Z^b = 0,$$

where $Z^a$ is a variable direction in the tangent $m$-plane at $p$.

Theorem 2 follows at once from (8), (13) and (11).

The hypercones at $p$ for the two distinct points $c (x^b_0 + c^k)$ and $d (x^b_0 + d^k)$ are the same if and only if a constant $\rho$ exists such that

$$\langle d^k - \rho c^k \rangle H^{\cdot \cdot k}_{eb} = (1 - \rho)'g_{eb}$$

is satisfied. If $d$ and $c$ have the same projection in $N'$ we have by (7)

$$\langle d^k - c^k \rangle H^{\cdot \cdot k}_{eb} = 0.$$

Therefore (14) will be satisfied by $\rho = 1$. Hence all the points in $\mathcal{N}$ with the same projection in $N'$ have the same hypercone $\mathcal{C}$.

This shows that we may confine our attention to the points in $N'$ for the consideration of the hypercone $\mathcal{C}$. Let this be done. Then (15) can no longer hold, and condition (14) cannot be satisfied by $\rho = 1$. Consequently, (14) may be put into the form (8) with

$$v^k = \langle d^k - \rho c^k \rangle / (1 - \rho).$$

Therefore if there exist two points in $N'$ with the same hypercone $\mathcal{C}$ then $\rho$ must be a semi-umbilical point. Conversely, at a semi-umbilical point the locus of the point in $N'$ whose hypercone $\mathcal{C}$ is the same as that of the point $c$ (distinct from $v$) is the straight line $cv$ minus the point $v$.

The hypercone (13) sustains an orthogonal ennuple, that is, contains $m$ mutually orthogonal generators if and only if

$$'g^{eb} \langle c_k H^{\cdot \cdot k}_{eb} - 'g_{eb} \rangle = 0,$$

that is, by (6) if

$$c_k M^k = 1.$$ 

Theorem 4 is an immediate consequence of this.

The hypercone (13) has a line of vertices if and only if

$$\text{Det} \langle c_k H^{\cdot \cdot k}_{eb} - 'g_{eb} \rangle = 0,$$

that is, if $c$ lies on the $K$-variety (9). This proves Theorem 5.