COUNTABLE CONNECTED SPACES

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Introduction. Let $\mathfrak{S}$ be the class of all countable and connected perfectly separable Hausdorff spaces containing more than one point. It is known that an $\mathfrak{S}$-space cannot be regular or compact. Urysohn, using a complicated identification of points, has constructed the first example of an $\mathfrak{S}$-space.\(^1\) Two $\mathfrak{S}$-spaces, $X$ and $X^*$, more simply constructed and not involving identifications, are presented here. The space $X^*$ is a connected subspace of $X$ and contains a dispersion point; that is, the subspace formed from $X^*$ by removing this one point is totally disconnected.

1. Sequences. The null sequence or any finite sequence of positive integers will hereafter be called more briefly a sequence. The null sequence or a sequence having an even number of elements is said to be even and a sequence having an odd number of elements is said to be odd. A sequence will usually be denoted by a lower case Greek letter: an arbitrary sequence by $\alpha$, $\beta$, or $\gamma$; an arbitrary even sequence by $\lambda$, $\mu$, or $\nu$; the null sequence by $\emptyset$. A positive integer will be denoted by a lower case italic letter (not $x$, $y$, or $z$), which may also serve to represent the sequence consisting of that single integer.

The relation $\alpha \geq i$ signifies that $a \geq i$ for every element $a$ of $\alpha$, and $\alpha < i$ that $a < i$ for every element $a$ of $\alpha$. The null sequence vacuously satisfies both $\emptyset \geq i$ and $\emptyset < i$.

The sequence formed by adjoining $\beta$ to the end of $\alpha$ is denoted by $\alpha \beta$.

Definition. The relation $\beta \supset \alpha$ is to mean that a sequence $\alpha'$ exists such that $\beta = \alpha \alpha'$ and $\alpha' \geq i$.

Some immediate consequences of the preceding definitions are:

1. $\alpha \supset \alpha$.
2. If $\beta \supset \alpha$ and $i \geq j$, then $\beta \supset \alpha$.
3. If $\gamma \supset \beta$ and $\beta \supset \alpha$, then $\gamma \supset \alpha$.
4. If $\gamma \supset \alpha$ and $\gamma \supset \beta$, then $\beta \supset \alpha$ or $\alpha \supset \beta$.

Proof. Let $\gamma \supset \alpha$ and $\gamma \supset \beta$; then sequences $\alpha'$, $\beta'$ exist such that

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\[ \gamma = \alpha \alpha', \alpha' \geq a, \text{ and } \gamma = \beta \beta', \beta' \geq b. \] Since \( \alpha \alpha' = \beta \beta' \), there exists a sequence \( \alpha'' \) such that \( \beta = \alpha \alpha'' \) or a sequence \( \beta'' \) such that \( \alpha = \beta \beta'' \).

If \( \beta = \alpha \alpha'' \), then \( \alpha \alpha' = \beta \beta' = \alpha \alpha' \beta' \); hence \( \alpha' = \alpha' \beta' \). But \( \alpha' \geq a \), so \( \alpha'' \geq a \) and consequently \( \beta \supseteq a \alpha \). Similarly, if \( \alpha = \beta \beta'' \), then \( \alpha \supseteq b \beta \).

2. Points in \( X \). The space \( X \) shall consist of two disjoint subsets: \( Y \), the set of all even sequences; and \( Z \), the set of all ordered pairs \( \{k, (\mu, \nu)\} \) composed of a positive integer \( k \) and a set \( (\mu, \nu) = (\nu, \mu) \) of even sequences \( \mu \) and \( \nu \). Hereafter a point \( \mu \) in \( Y \) will be denoted by \( y(\mu) \) and a point \( \{k, (\mu, \nu)\} \) in \( Z \) by \( z_k(\mu, \nu) \). Evidently \( X \) is countable.

The neighborhoods in \( X \) will be formed from certain subsets \( Y_i(\alpha) \) of \( Y \), defined for every positive integer \( i \) and every sequence \( \alpha \).

Definition. \( Y_i(\alpha) \) is the set of all points \( y(\mu) \) such that \( \mu \supseteq \alpha \).

Some properties of these sets are:

2.1. \( y(\mu) \in Y_i(\mu) \).

2.2. If \( j \geq i \), then \( Y_j(\alpha) \subseteq Y_i(\alpha) \).

2.3. If \( y(\mu) \in Y_i(\alpha) \), then \( Y_i(\mu) \subseteq Y_i(\alpha) \).

2.4. \( Y_\alpha(\alpha) Y_\beta(\beta) \neq 0 \) is equivalent to: \( \beta \supseteq \alpha \alpha \) or \( \alpha \supseteq b \beta \).

Proof. If the set \( Y_\alpha(\alpha) Y_\beta(\beta) \) contains a point \( y(\mu) \), then \( \mu \supseteq \alpha \alpha \) and \( \mu \supseteq b \beta \); therefore, \( \beta \supseteq \alpha \alpha \) or \( \alpha \supseteq b \beta \).

Now, if \( \beta \supseteq \alpha \alpha \), define \( m = \max (a, b) \), \( \nu = \beta \) if \( \beta \) is even, \( \nu = \beta m \) if \( \beta \) is odd. Thus \( \nu \) is even and \( \nu \supseteq b \beta \) so \( y(\nu) \in Y_\beta(\beta) \). Moreover \( \nu \supseteq \alpha \alpha \), hence \( \nu \supseteq \alpha \alpha \) so \( y(\nu) \in Y_\alpha(\alpha) \). Therefore \( Y_\alpha(\alpha) Y_\beta(\beta) \neq 0 \). Similarly \( Y_\alpha(\alpha) Y_\beta(\beta) \neq 0 \) if \( \alpha \supseteq b \beta \).

Corollary. If \( \alpha \neq \beta \) and \( \alpha \beta < i \), then \( Y_i(\alpha) Y_i(\beta) = 0 \).

To every point \( z = z_k(\mu, \nu) \) a unique positive integer \( q(z) = q_k(\mu, \nu) \) is assigned as follows. The set of all sets \( (\mu, \nu) \) of even sequences \( \mu \) and \( \nu \), being countable and infinite, can be mapped onto the set of positive integral primes by some 1-1 mapping \( p(\mu, \nu) \). Define

\[ q_k(\mu, \nu) = [p(\mu, \nu)]^k \]

According to the unique factorization theorem of arithmetic, \( q \) is a 1-1 mapping of the point set \( Z \) onto a subset of the positive integers. Moreover, since the infinite sequence of positive integers \( q_k(\mu, \nu) \) for \( k = 1, 2, \cdots \) is strictly increasing, \( q_k(\mu, \nu) \rightarrow \infty \) as \( k \rightarrow \infty \).
3. **Neighborhoods in X.** For every point \( x \) in \( X \) and every positive integer \( i \), a neighborhood \( V_x^i \) of \( x \) is now defined.

**Definition.**

\[
V_{i,y}(\mu) = Y_i(\mu); \\
V_{i,\beta}(\mu, \nu) = \beta_k(\mu, \nu) + Y_i(\mu q) + Y_i(\nu q), \quad q = q_k(\mu, \nu).
\]

Under this definition of neighborhood \( X \) forms a Hausdorff topological space; that is, \( X \) satisfies the following neighborhood axioms.

**Axiom 1.** To every point \( x \) in \( X \) there corresponds at least one neighborhood of \( x \); every neighborhood of \( x \) contains \( x \) by 2.1 or by definition.

**Axiom 2.** If \( V_x \) and \( V_{x'} \) are two neighborhoods of \( x \), a neighborhood \( V_m \) of \( x \) exists such that \( V_m \subset V_x V_{x'} \). Indeed, if \( m = \max(i, j) \), then \( V_m = V_x V_{x'} \) by 2.2.

**Axiom 3.** If \( V_x \) contains a point \( y(\mu) \), there exists a neighborhood of \( y(\mu) \) contained in \( V_x \). By 2.3 such a neighborhood is \( V_{i,y}(\mu) \).

**Axiom 4.** Every two distinct points \( x, x' \) in \( X \) are Hausdorff- or \( H \)-separable; that is, there exist neighborhoods \( V_x \) of \( x \) and \( V_{x'} \) of \( x' \) such that \( V_x V_{x'} = 0 \). The intersection \( V_x V_{x'} \) can be reduced to the sum of at most four intersections, each of the form \( Y_i(\alpha) Y_i(\alpha') \). If \( \alpha, \alpha' \) are both even, then \( \alpha \neq \alpha' \) since \( x \neq x' \). And also \( \alpha \neq \alpha' \), if \( \alpha, \alpha' \) are both odd; for then even sequences \( \alpha, \alpha' \) and positive integers \( q, q' \) exist such that \( \alpha = \mu q, \alpha' = \mu' q' \), and, since \( x \neq x' \), \( q \neq q' \). Thus, according to the corollary of 2.4, \( Y_i(\alpha) Y_i(\alpha') = 0 \) when \( i \) is chosen so that \( \alpha \alpha' < i \). An integer \( i \) then exists for which \( V_x V_{x'} = 0 \).

Thus \( X \) is a nondegenerate countable Hausdorff space. Evidently \( X \) is also perfectly separable.

4. **Connectedness of \( X \).** Two distinct points \( x, x' \) in a space \( E \) are said to be \( H \)-separable provided neighborhoods \( V \) of \( x \) and \( V' \) of \( x' \) exist such that \( V V' = 0 \); otherwise, the points \( x, x' \) are said to be \( H \)-inseparable. A single point is also said to be \( H \)-inseparable if it is \( H \)-inseparable with every other point in \( E \).

A space \( E \) containing an \( H \)-inseparable point \( x \) is connected; for otherwise \( E \) could be covered by two non-null disjoint isolated (open and closed) sets \( V, V' \), one of which contains \( x \); but this would imply the contradiction

\[
0 = V V' = V V' \neq 0.
\]

Moreover, if \( E \) is a Hausdorff space, then no point of \( E \) satisfies the regularity axiom, or, more briefly, is regular. For let \( x' \) be any point
in $E$ distinct from $x$. Since $x, x'$ are $H$-separable in $E$, there exist disjoint neighborhoods $V$ of $x$ and $V'$ of $x'$; consequently

$$VV' = 0 = VV'.$$

If $x$ were a regular point of $E$, then a neighborhood $U$ of $x$ would exist such that $V \supset U$, so

$$0 = VV' \supset UV' \neq 0.$$

Similarly, if $x'$ were a regular point of $E$, then a neighborhood $U'$ of $x'$ would exist such that $V' \supset U'$, so

$$0 = VV' \supset UV' \neq 0.$$

By considering the sets $Y_i(\alpha)$ every point in the space $X$ is now shown to be $H$-inseparable. Hence $X$ is connected and no point of $X$ is regular.

**Definition.** $Z_i(\alpha)$ is the set of all points $z_k(\mu, v)$ such that $\mu \supset \alpha$ or $\nu \supset \alpha$, $q = q_k(\mu, v)$.

4.1. $Y_i(\alpha) = Y_i(\alpha) + Z_i(\alpha)$.

**Proof.** The following equivalent statements show that $Y_i(\alpha) = Y_i(\alpha)$:

- $y(\mu) \in Y_i(\alpha)$.
- For all $j$: $Y_j(\mu) Y_i(\alpha) \neq 0$.
- For all $j$: $Y_j(\mu) Y_i(\alpha) \neq 0$.
- For all $j$: $\alpha \supset \mu$ or $\mu \supset \alpha$.
- $\mu \supset \alpha$.
- $y(\mu) \in Y_i(\alpha)$.

The following equivalent statements show that $Z_i(\alpha) = Z_i(\alpha)$, where $q_k(\mu, v)$ has been abbreviated to $q$:

- $z_k(\mu, v) \in Z_i(\alpha)$.
- For all $j$: $Y_j z_k(\mu, v) Y_i(\alpha) \neq 0$.
- For all $j$: $[Y_i(\mu) + Y_i(\nu)] Y_i(\alpha) \neq 0$.
- For all $j$: $Y_i(\mu) Y_i(\alpha) \neq 0$ or $Y_i(\nu) Y_i(\alpha) \neq 0$.
- For all $j$: $\mu \supset \nu$ or $\mu \supset \nu$ or $\nu \supset \alpha$.
- $z_k(\mu, v) \in Z_i(\alpha)$.

4.2. $Z_\alpha(\alpha) Z_\alpha(\beta) \neq 0$; hence every two distinct points in $X$ are $H$-inseparable.

**Proof.** Evidently there exist even sequences $\mu, \nu$ such that $\mu \supset \alpha$ and $\nu \supset \beta$. And since $q_k(\mu, v) \to \infty$ as $k \to \infty$ a positive integer $k$ exists for which
q = q_k(\mu, \nu) \geq \max (a, b).

Therefore \mu q \supset \alpha and \nu q \supset \beta; so \varepsilon_k(\mu, \nu) \in Z_\alpha(\alpha)Z_\beta(\beta).

Thus X is an \mathcal{S}-space whose every point is \mathcal{H}-inseparable.

5. The space X*. Let X* be the relative subspace of X formed by removing from X all points \varepsilon_k(\mu, \nu) except those of the form \varepsilon_k(\mu, o), \mu \neq o. Notice that every X*-neighborhood of a point in X* is also an X-neighborhood of that point. The argument of 4.2 shows that the set \mathcal{Z}_\alpha(\alpha)Z_\beta(o) contains a point of X*. The point y(o) is then an \mathcal{H}-inseparable point of X*. Thus X*, being a nondegenerate connected subspace of an \mathcal{S}-space, is also an \mathcal{S}-space.

6. The space X**. Let X** be the relative subspace of X* formed by removing from X* the single point y(o). This point is a dispersion point of X*; for the following recursive construction of isolated subsets in the space X** shows that X** is totally disconnected.

**DEFINITION.** For every non-null even sequence \lambda and every positive integer i such that \lambda < i let

\[ X_i(\lambda) = \sum_{n=1}^\infty [V^n_i(\lambda) + Z^n_i(\lambda)], \]

the sets \( V^n_i(\lambda) \) and \( Z^n_i(\lambda) \) being recursively defined as follows:

- \( V^n_i(\lambda) \) is the set of all points y(\mu) such that \( \mu \supset \alpha^n \), where \( \alpha^n = \lambda \) if \( n = 1 \), and \( \alpha^n = g(z) \) for some \( z \in Z^{n-1}_i(\lambda) \) if \( n > 1 \);

- \( Z^n_i(\lambda) \) is the set of all points \( z = \varepsilon_k(\mu, o) \) such that \( y(\mu) \in V^n_i(\lambda) \) and \( g(z) \geq i \).

6.1. \( V_i x \subset X_i(\lambda) \) for all \( x \in X_i(\lambda) \); hence \( X_i(\lambda) \) is open in \( X** \).

**PROOF.** Let \( y(\mu) \in V^n_i(\lambda) \); then \( \mu \supset \alpha^n \). If \( y(\nu) \in V_i y(\mu) \), then \( \nu \supset \mu \supset \alpha^n \), so \( y(\nu) \in V^n_i(\lambda) \).

Let \( z = \varepsilon_k(\mu, o) \in Z^n_i(\lambda) \); then \( \mu \supset \alpha^n \) and \( g(z) \geq i \). If \( y(\nu) \in V_i z \), then \( \nu \supset \mu g(z) \) or \( \nu \supset g(z) \). Now \( \nu \supset \mu g(z) \), \( g(z) \geq i \), implies that \( \nu \supset \mu \supset \alpha^n \) and hence that \( y(\nu) \in V^n_i(\lambda) \). And \( \nu \supset g(z) \), \( z \in Z^n_i(\lambda) \), implies that \( y(\nu) \in V^n_i(\lambda) \).

6.2. \( V_i x X_i(\lambda) = 0 \) for all \( x \in X_i(\lambda) \); hence \( X_i(\lambda) \) is closed in \( X** \).

**PROOF.** Let \( y(\mu) \in X_i(\lambda) \). Suppose the set \( V_i y(\mu) X_i(\lambda) \) contains a point \( y(\nu) \); then \( \nu \supset \mu \) and \( \nu \supset \alpha^n \). Therefore \( \mu \supset \alpha^n \) or \( \alpha^n \supset \mu \). Now \( \alpha^n \supset \mu \), since \( \alpha^n = \lambda < i \) and since \( \alpha^n \) is a single integer if \( n > 1 \). Hence \( \mu \supset \alpha^n \), so \( y(\mu) \in V^n_i(\lambda) \)—a contradiction.

Let \( z = \varepsilon_k(\mu, o) \in X_i(\lambda) \). Suppose the set \( V_i z X_i(\lambda) \) contains a point
\( y(\nu) \); then \( \nu \supseteq \mu q(\varepsilon) \) or \( \nu \supseteq q(\nu) \), and \( \nu \supseteq \alpha^n \). Therefore

\[ \forall n: \alpha^n \supseteq \mu q(\varepsilon) \text{ or } \mu q(\varepsilon) \supseteq \alpha^n \text{ or } \alpha^n \supseteq q(\varepsilon) \text{ or } q(\varepsilon) \supseteq \alpha^n. \]

Now \( \lambda \neq o, \lambda < i \), and \( \lambda = \lambda^1 \); \( \forall^1 \) then reduces to \( \mu q(\varepsilon) \supseteq \lambda \); so \( \mu \supseteq \lambda \), \( q(\varepsilon) \geq i \), and consequently \( z \in Z^1_q(\lambda) \)—a contradiction. If \( n > 1 \), then \( \alpha^n = q(\varepsilon') \geq i \) for some \( \varepsilon' \in Z^{n-1}_i(\lambda) \), so \( \forall^n \) reduces to: \( \mu q(\varepsilon) \supseteq q(\varepsilon') \), \( \mu \neq o \); or \( q(\varepsilon) = q(\varepsilon') \). Now \( \mu q(\varepsilon) \supseteq q(\varepsilon') \), \( \mu \neq o \), implies that \( \mu \supseteq q(\varepsilon') \), \( q(\varepsilon) \geq i \), and hence that \( z \in Z^1_q(\lambda) \)—a contradiction. And \( q(\varepsilon) = q(\varepsilon') \) implies that \( z = z' \in Z^{n-1}_i(\lambda) \)—also a contradiction.

The sets \( X_i(\lambda) \) are then isolated subsets of \( X^{**} \) for \( \lambda \neq o, \lambda < i \). Notice that

\[
\begin{align*}
x &= y(\lambda) \in X_i(\lambda), \\
x &= y(\lambda') \in X_i(\lambda) & \text{if } \lambda' \neq \lambda \text{ and } \lambda' < i, \\
x &= s_k(\lambda, o) \in X_i(\lambda) & \text{if } q(x) \geq i, \\
x' &\in X_i(\lambda) & \text{if } x' \in Z \text{ and } q(x') < i.
\end{align*}
\]

Now there exists for any two distinct points \( x, x' \) in \( X^{**} \) an isolated set \( X_i(\lambda) \) containing \( x \) but not \( x' \): if \( x = y(\lambda), x' = y(\lambda') \), choose \( i \) so that \( \lambda \lambda' < i \); if \( x = y(\lambda) \) and \( x' \in Z \), choose \( i \) so that \( \lambda q(x') < i \); and if \( x = s_k(\lambda, o) \) and \( x' \in Z \), choose \( i = q(x) \), then \( q(x') < i \) and \( \lambda < i \), since it may be assumed that \( q(x') < q(x) \) and since the mapping \( p \) can be selected so that \( \mu \nu < p(\mu, \nu) \).

Thus the space \( X^{**} \) is totally disconnected. In particular, every two distinct points in \( X^{**} \) are \( \overline{H} \)-separable; hence \( y(o) \) is the only \( \overline{H} \)-inseparable point of \( X^* \).

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