ON SOME ASYMPTOTIC FORMULAS IN THE
THEORY OF PARTITIONS

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Let \( p(n) \) denote the number of unrestricted partitions of \( n \). \( p_k(n) \) denotes the number of partitions of \( n \) into precisely \( k \) summands, or what is the same into partitions whose largest summand is \( k \). Auluck, Chowla and Gupta\(^1\) announced the following conjecture:

For \( n \) fixed let \( p_{k_0}(n) \) be the greatest \( p_k(n) \); that is, \( p_{k_0}(n) \geq p_k(n) \). Then

\[
(1) \quad k_0 \sim c^{-1}n^{1/2} \log n, \quad c = \pi(2/3)^{1/2}.
\]

They prove that

\[
n^{1/2} < k_0 < (1 + \delta)c^{-1}n^{1/2} \log n
\]

for every \( \delta > 0 \) if \( n \) is sufficiently large.

In the present note we shall prove (1). In fact we shall prove that

\[
(2) \quad k_0 = c^{-1}n^{1/2} \log n + an^{1/2} + o(n^{1/2}) \quad \text{where} \quad c/2 = e^{-\alpha/2}.
\]

They also conjectured that for \( k_1 < k_2 \leq k_0, \ p_{k_1}(n) \leq p_{k_2}(n) \) and for \( k_0 < k_1 < k_2, \ p_{k_1}(n) < p_{k_2}(n) \). They verify this conjecture for \( n \leq 32 \).

Recently Todd\(^2\) published a table of all the \( p_k(n) \) for \( n \leq 100 \), and it is easy to verify the conjecture for \( n \leq 100 \). I am unable to prove or disprove this conjecture. They also remark that \( p_{k_0}(n) \) differs from \( c^{-1}n^{1/2} \log n \) by less than 1 for \( n \leq 32 \); (2) shows that for large \( n \) the difference tends to infinity.

Lehner and I\(^3\) proved that if we denote \( P_k(n) = \sum_{r \leq k} p_r(n) \) then for \( k = c^{-1}n^{1/2} \log n + \lambda n^{1/2} \) we have the asymptotic formula

\[
(3) \quad P_k(n)/p(n) = (1 + o(1)) \exp (- (2/c)e^{-\alpha/2}).
\]

In proving (2) we shall use (3) a great deal, we shall also use the well known asymptotic formula

\[
(4) \quad p(n) = (1 + o(1))(1/4 \cdot 3^{1/2}n) \exp (cn^{1/2}).
\]

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Let \( f(n) \) tend to infinity arbitrarily slowly; we easily obtain from (3) that for \( k_1 = [c^{-1} n^{1/2} \log n + f(n) n^{1/2}] \), \( k_2 = [c^{-1} n^{1/2} \log n - f(n) n^{1/2}] \).

\[
\frac{1}{p(n)} (P_{k_1}(n) - P_{k_2}(n)) \to 1 \quad \text{as} \quad n \to \infty.
\]

We immediately obtain from (4) and (5) that for some \( k_2 < k_3 < k_1 \)

\[
p_{k_3}(n) > c_1 p(n) / n^{1/2} > (c_2 / n^{3/2}) \exp (c n^{1/2}).
\]

\( c_1, c_2, \cdots \) denote absolute constants. Thus

\[
p_{k_0}(n) \geq p_{k_2}(n) > (c_2 / n^{3/2}) \exp (c n^{1/2}).
\]

Now we show that for sufficiently large \( c_2 \)

\[
k_0 < c^{-1} n^{1/2} \log n + c_3 n^{1/2}.
\]

Let \( k_4 \geq c^{-1} n^{1/2} \log n + c_3 n^{1/2} \). It clearly follows from the definition of \( p_4(n) \) and \( P_k(n) \) that \( p_{k_4}(n) = P_{k_4}(n - k_4) < p(n - k_4) \). Thus from (4)

\[
p_{k_4}(n) < (c_4 / n) \exp (c(n - k_4)^{1/2}) < (c_4 / n) \exp (c(n^{1/2} - k_4 / 2 n^{1/2})
\]

\(< (c_4 / n) \exp (c(n^{1/2} - \log n / 2 - c_5 / 2)) \)

\(< (c_2 / n^{3/2}) \exp (c n^{1/2}) < p_{k_0}(n)
\]

for sufficiently large \( c_5 \), and this proves (8).

Next we prove that for sufficiently large \( c_6 \)

\[
k_0 > c^{-1} n^{1/2} \log n - c_6 n^{1/2}.
\]

Suppose (9) does not hold. We obtain from (7) that for some \( k_0 < c^{-1} n^{1/2} \log n - c_6 n^{1/2} \)

\[
p_{k_0}(n) > (c_2 / n^{3/2}) \exp (c n^{1/2}).
\]

We shall show that (10) leads to a contradiction. First we show that

\[
p_k(n) \leq p_{k+i}(n + j)
\]

for \( j \geq i \).

We have

\[
p_k(n) \leq p_{k+i}(n + i) \leq p_{k+i}(n + j).
\]

The first inequality of (12) we obtain by mapping the partition \( a_i + \cdots + k \) of \( p_k(n) \) into \( a_i + \cdots + (k + i) \) which belongs to \( p_{k+i}(n + i) \), the second part we obtain by adding \( j - i \) 1's to every partition of \( p_{k+i}(n + i) \); this proves (11).

Put \( [n^{1/2}] = b \); we have from (10) and (11) for \( 0 \leq i \leq b \)

\[
p_{k_0+i}(n + b) \geq p_{k_0+i}(n) > (c_2 / n^{3/2}) \exp (c n^{1/2})
\]

\(< (c_6 / n^{3/2}) \exp (c(n + b)^{1/2}).
\]

Thus
\[ \sum_{i=0}^{b} p_{k_0+i}(n + b) > (c_8/n) \exp \left( (c(n + b)^{1/2}) \right). \]

Now we obtain from (5) that for every \( \varepsilon \) and sufficiently large \( c_8 \) and \( n \)
\[ \sum_{k > k_0 + b} p_k(n + b) > (1 - \varepsilon)p(n + b). \]

The proof of (14) follows immediately from the fact that \( k_0 + b < c^{-1}n^{1/2} \log n - (c_8 - 1)n^{1/2} \), thus (5) can be applied. From (13) and (14) we have
\[ p(n + b) > \sum_{i=0}^{b} p_{k_0+i}(n + b) + \sum_{k > k_0 + b} p_k(n + b) \]
\[ > (1 - \varepsilon)p(n + b) + (c_8/n) \exp \left( (c(n + b)^{1/2}) \right). \]

Thus
\[ \varepsilon p(n + b) > (c_8/n) \exp \left( (c(n + b)^{1/2}) \right), \]
which contradicts (4); this proves (9).

We now know from (8) and (9) that \( k_0 \) has to satisfy
\[ c^{-1}n^{1/2} \log n - c_8n^{1/2} < k_0 < c^{-1}n^{1/2} \log n + c_8n^{1/2}. \]

Put
\[ k_0 = c^{-1}n^{1/2} \log n + xn^{1/2}. \]

We obtain from (3) and (4) that
\[ p_{k_0}(n) = P_{k_0}(n - k_0) \]
\[ = (1 + o(1))p(n)n^{-1/2} \exp \left( - cx/2 - (2/c) \exp \left( - cx/2 \right) \right). \]

The right side is maximal if \( c/2 = \exp(-cx/2) \), which completes the proof of (2).

We immediately obtain from (2) and (15) that
\[ \lim p_{k_0}(n)n^{1/2}/p(n) = \exp \left( - ca/2 - (2/c) \exp \left( - ax/2 \right) \right). \]

It would be easy to sharpen the error term \( o(n^{1/2}) \) in (2) by getting an error term in (3), but it seems very hard to get a sufficiently good inequality to prove the conjecture of Auluck, Chowla and Gupta.

Denote by \( Q(n) \) the number of partitions of \( n \) into unequal parts. \( Q_k(n) \) denotes the number of partitions of \( n \) into precisely \( k \) unequal parts. Define \( k_0 \) by
\[ Q_{k_0}(n) \geq Q_k(n). \]

4 This formula is due to Auluck, Chowla and Gupta (ibid).
It has been conjectured that for \( k_1 < k_2 \leq k_0, Q_{k_2}(n) < Q_{k_1}(n) \) and for \( k_0 < k_1 < k_2, Q_{k_2}(n) \geq Q_{k_1}(n) \). This conjecture we can not decide. But by using Theorem 3.3 of our paper with Lehner we can show that

\[
k_0 = 2 \log 2n^{1/2} / \pi(1/3)^{1/2} + d n^{1/4} + o(n^{1/4})
\]

for a certain constant \( d \). Also

\[
\lim n^{1/4} Q_{k_0}(n)/Q(n) \to e, \text{ for a certain constant } e.
\]

We do not discuss the proofs. They are similar but slightly more complicated than the proof of (2).

It would be interesting to get an asymptotic formula for \( p_k(n) \) and \( Q_k(n) \). Perhaps the first step would be to get an asymptotic formula for \( \log p_k(n) \). It is easy to see that for \( k = o(n^{1/2}) \)

\[
\log p_k(n) = o(n^{1/2})
\]

and if \( k/n^{1/2} \to \infty \)

\[
\log p_k(n)/\log \pi(n) \to 1.
\]

The proofs can be obtained easily by simple Tauberian theorems.