INEQUALITIES CONNECTING SOLUTIONS OF CREMONA'S EQUATIONS

G. B. HUFF

1. Introduction. Let a complete and regular linear system $\Sigma_{p,d}$ of plane curves of dimension $d$, the genus of the general curve being $p$, be determined by its order $x_0$, and its multiplicities $x_1, \ldots, x_p$ at a set of $p$ general base points. $x = (x_0; x_1, \ldots, x_p)$ is called the characteristic of $\Sigma_{p,d}$ and satisfies Cremona's equations:

\begin{align*}
\frac{x_0^2}{2} - x_1 - x_2 - \cdots - x_p \equiv (xx) &= d + p - 1, \\
3x_0 - x_1 - x_2 - \cdots - x_p \equiv (lx) &= d - p + 1.
\end{align*}

On the other hand, an integer solution $x$ of (1) may or may not determine a linear system. If an $x$ does determine a $\Sigma_{p,d}$, it is said to be proper. In this definition is included the usual convention that $(0; -1, 0, \ldots, 0)$ is a proper characteristic of the set of directions at a base point $[1]$.\(^1\)

If a system $\Sigma_{p,d}$ of characteristic $x$ is subjected to a Cremona transformation $C$ with $F$-points at the base points of $\Sigma$, $\Sigma \rightarrow \Sigma_{p,d}$ whose characteristic $x'$ at the $F$-points of $C^{-1}$ is given by:

\begin{align*}
L: \quad x'_0 &= (cx) = c_0x_0 - c_1x_1 - c_2x_2 - \cdots - c_px_p, \\
x'_i &= (f^i x) = f^i_0x_0 - f^i_1x_1 - f^i_2x_2 - \cdots - f^i_px_p, \\
&\quad i = 1, 2, \ldots, p.
\end{align*}

Here $c$ is the characteristic of the homaloidal net of $C^{-1}$ and the $f^i$ are the characteristics of the $P$-curves of this net. Thus proper characteristics $c$ of $p=0, d=2$ and proper characteristics $f$ of $p=d=0$ play a central role in the theory and will be prominent in this article. The collection of all transformations $L$ for a given $p$ forms a group, $G_p$. $G_p$ is generated by transformations $L$ for which $c$ is of type $(2; 1110 \cdots 0)$, and for any $L \in G_p$ the forms $(xx), (lx)$ and $(xy)$ are invariant.

In this paper attention is restricted to characteristics of $x_0 > 0$, and $p \geq 0$ and $d \geq 0$. We shall designate this as property A and obtain inequalities implied by (1) and property A. The inequalities are interesting in themselves and lead to a criterion for distinguishing proper characteristics.

\(^1\) Numbers in brackets refer to the references cited at the end of the paper.
2. Inequalities involving the characteristics of homaloidal nets.

**Theorem 1.** If \( x \) has property A, then \( 2x_0 - x_1 - x_2 - x_3 \geq 0 \). Moreover, the equals signs hold only for \( p = d = 0; x = (1; 110), (1; 101) \) or \( (1; 011) \).

Since \((xx) = d + p - 1 \geq -1\), it may be shown that \( x_0 \geq x_i, i = 1, \ldots, p \). Indeed, set \( x_i = x_0 - a \) in \((xx) \geq -1:\)

\[
2ax_0 - a^2 - x_1^2 - \cdots - x_{i-1}^2 - x_{i+1}^2 - \cdots - x_p^2 \geq -1,
\]
or

\[
a(2x_0 - a) \geq -1.
\]

Since \( x_0 \) is a positive integer, \( a \) may not be negative. Thus the integers \( a_1, a_2, a_3 \) in \( x_1 = x_0 - a_1, x_2 = x_0 - a_2, x_3 = x_0 - a_3 \) are non-negative. Substituting these in the quadratic relation yields:

\[-2x_0 + 2x_0(a_1 + a_2 + a_3) - a_1^2 - a_2^2 - a_3^2 - x_4^2 - \cdots - x_p^2 \geq -1.
\]

Now \( a_1, a_2, a_3, x_4, \ldots, x_p \) cannot all vanish, for this would imply that \(-2x_0^2 \geq -1\). Thus:

\[
2x_0(a_1 + a_2 + a_3) - 2x_0^2 > 1
\]
or

\[
a_1 + a_2 + a_3 - x_0 > -1/2x_0.
\]

It follows that \( a_1 + a_2 + a_3 - x_0 \geq 0 \) and thus that

\[
2x_0 - (x_0 - a_1) - (x_0 - a_2) - (x_0 - a_3) \geq 0.
\]

If \( x \) is a characteristic with property A and \( 2x_0 - x_1 - x_2 - x_3 = 0 \), then the image of \( x \) under

\[
x_0' = 2x_0 - x_1 - x_2 - x_3,
\]

\[
A_{123}: \ x_i' = x_i + (x_0 - x_1 - x_2 - x_3), \quad i = 1, 2, 3,
\]

\[
x_j' = x_j, \quad j = 4, \ldots, p,
\]

has \( x_0' = 0 \) and satisfies the same Cremona equations. Thus

\[
- x_1'^2 - x_2'^2 - \cdots - x_p'^2 = d + p - 1,
\]

\[
- x_1' - x_2' - \cdots - x_p' = d - p + 1.
\]

This is possible only for \( d, p = 0, 0; 1, 0 \) and 0, 1. A canvass of the cases reveals that \( d, p = 0 \) and \( x' = (0; -1 0 \cdots 0) \) comprise all possibilities. Thus \( x = (1; 110), (1; 101), (1; 011) \) are the only values of \( x \) for which the equals sign holds.
Since \((2; 1110 \cdots 0)\) is the characteristic of a homaloidal net of conics, the form of the inequality clearly suggests the following generalization:

**Theorem 2.** If \(x\) has property \(A\) and \(c\) is the characteristic of a homaloidal net, then \((cx) \geq 0\). Moreover, the equals sign holds only for the characteristics of the principal curves of the homaloidal net.

Consider first characteristics \(x\) of \(p+d > 0\). In this case, Theorem 1 asserts that any \(x'\) obtained from \(x\) under \(A_{ijk}\) has \(x'_0 > 0\). Since \(c\) is the characteristic of a homaloidal net, \(c\) is the image of \((1; 0, 0, \cdots, 0)\) under a sequence of transformations of the form \(A_{ijk}\). Let \(x \rightarrow x'\) under the sequence that sends \(c \rightarrow c' = (1; 0, \cdots, 0)\). Since \(x'_0 > 0\), it follows that \((c'x') > 0\). Thus \((cx) > 0\), for this bilinear relation is invariant under \(G_p\).

If \(p=0\), \(d=0\), a modification of the argument is required since in this case \(x'_0\) might vanish under some \(A_{ijk}\). But in this case \(x\) is by Theorem 1 a proper characteristic. Thus an improper characteristic \(x\) of \(p=d=0\) always goes into a characteristic of \(x'_0 > 0\) under \(A_{ijk}\) and the argument above applies. For proper characteristics \(x\) of \(p=d=0\), it is clear that \((cx) \geq 0\), else the rational curve would have too many intersections with the homaloidal net. If \((cx)=0\), \(x\) is the characteristic of a rational curve meeting the curves of the net only at the base points, and hence is the characteristic of a principal curve of the net.

3. **Inequalities involving characteristics of rational curves.**

**Lemma.** If \(x\) has property \(A\) and \(x^*\) denotes the same characteristic with \(x_p\) deleted, then \(x^*\) has property \(A\).

A simple computation yields for \(p', d'\) of \(x^*:\)

\[
d' = d + x_p(x_p + 1)/2, \quad p' - 1 = p - 1 + x_p(x_p - 1)/2.
\]

Since \(x_p(x_p+1)/2\) and \(x_p(x_p-1)/2\) are non-negative functions of the integer \(x_p\), the conclusion follows.

**Theorem 3.** If \(x\) has property \(A\) and \(p+d > 0\), and \(f\) is a proper characteristic of \(p=d=0\) and \((fx) < 0\), then \(x_0 > f_0\).

Since \(f\) is proper, there is [2] an \(L \subseteq G_p\) such that \(\tilde{f} = L(f) = (0; 0, \cdots, 0, -1)\). \(\tilde{x} = L(x)\) has \(\tilde{x}_0 > 0\) by Theorem 2 and \((f\tilde{x}) = (f\tilde{x}) < 0\). But \((f\tilde{x}) = \tilde{x}_0 < 0\). Thus \(\tilde{x}\) may be written in the form

\[
\tilde{x} = \tilde{x}^* + k\tilde{f},
\]

where \(k\) is a positive integer, and \(\tilde{x}^*\) is \(\tilde{x}\) with \(\tilde{x}_p\) deleted. Now consider the image of \(\tilde{x}\) under \(L^{-1}\).
\[ L^{-1}(x) = L^{-1}(\bar{x}^* + kf) = L^{-1}(\bar{x}^*) + kL^{-1}(f), \]

or
\[ x = L^{-1}(\bar{x}^*) + kf. \]

Now \( \bar{x}^* \) has \( \bar{x}_0^* > 0 \), and \( p' + d' > 0 \) by the lemma. Hence by Theorem 2 its image \( (\bar{x}^*)' \) has \( (\bar{x}_0^*)' > 0 \). Since
\[ x_0 = (\bar{x}_0^*)' + kf_0, \]
it follows that \( x_0 > f_0 \).

**Theorem 4.** *If \( x \) has property A and \( p + d > 0 \), and \( f \) is a proper characteristic such that \( p = d = 0 \) and \( x_0 \equiv f_0 \), then \( (fx) \equiv 0 \).*

Theorem 4 follows from Theorem 3 by formal reasoning and offers a generalization of a property of proper characteristics. For if \( x \) is a proper characteristic, \( (fx) \equiv 0 \) follows from the fact that the curves of the system may not have more than \( f_0 x_0 \) intersections with the irreducible rational curve associated with \( f \). The significance of the theorem is that all characteristics \( f_0 \equiv x_0 > 0 \), \( p \equiv 0 \), \( d \equiv 0 \), \( p + d > 0 \) must enjoy this same property.

There are examples of characteristics with property A and \( p + d > 0 \) which even have \( x_i > 0 \), \( i = 1, \ldots, \rho \), for which there is an \( f \) of \( f_0 < x_0 \) such that \( (fx) < 0 \). An early example is \((5; 3^11^6)\) and \((1; 1^20^6)\).

4. Applications.

**Theorem 5.** *Let \( x \) be a characteristic of property A and \( p + d > 0 \), such that \( (fx) \equiv 0 \) for all proper \( f \) of \( p = d = 0 \) and \( f_0 < x_0 \); then \( x_i \equiv 0 \) and, moreover, if \( x' \) is the image of \( x \) under any \( L \in G_\rho \), then \( x'_i > 0 \) and \( x'_i \equiv 0 \) for \( i = 1, \ldots, \rho \).*

Since \( f = (0; 0^01^1 - 1) \) is a proper \( f \) of \( f_0 = 0 < x_0 \) and \( (fx) \equiv 0 \), it follows that \( x_i \equiv 0 \). By Theorem 4, \( (fx) \equiv 0 \) for all proper \( f \), \( p = d = 0 \) of \( f_0 \equiv x_0 \). Then \( (fx) \equiv 0 \) for all proper \( f \). These characteristics \( f \) are simply permuted by any \( L \in G_\rho \). Thus if \( x' = L(x) \), it follows that \( (fx') \equiv 0 \) for all proper \( f \). Since these include \((0; 0^01^1 - 1)\), it follows as before that \( x'_i \equiv 0 \). Theorem 2 asserts that \( x'_i > 0 \).

The following important result is now easily established:

**Theorem 6.** *Let \( c \) be a solution of \((1)\) for \( p = 0, d = 2, c_0 > 0 \) such that \( (fc) \equiv 0 \) for all proper \( f \) of \( p = d = 0 \) and \( f_0 < c_0 \), then \( c \) is the characteristic of a homaloidal net.*

As before, \( c_i \equiv 0 \) and it is known \([3]\) that in such a case \( c_0 - c_1 - c_2 - c_3 < 0 \) if \( c_1, c_2, c_3 \) are the greatest of the numbers \( c_i \) and \( c_0 > 1 \). Thus
under $A_{123}$, $c \rightarrow c'$ of $c_0' < c_0$ and by Theorem 5, $c_i' \geq 0, i = 1, 2, \cdots, p$. Thus this reduction may be continued until $c_i'' = 1$, in which case $c'' = (1; 0, \cdots, 0)$. Under the given hypotheses, $c$ is then the image of $(1; 0, \cdots, 0)$ under some $L \subseteq G$, and must be proper.

This result has been conjectured much earlier and indeed was proved [4] by the writer, but the proof given on that occasion was quite elusive and unsatisfactory. Fragmentary results indicate that Theorem 5 has other important applications to cases where a generalization of Noether’s inequality is possible. It would be desirable to avoid the restriction $p + d > 0$. It is possible that Theorem 5 might still be true if one removed this restriction and added at the end of the theorem “or else $x'$ is of the type $(0; 0, \cdots, 0, -1)$.”

References


University of Texas