ON THE THEOREM OF FEJÉR-RIESZ

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1. Statement of results. Let

\[ f(z) = a_0 + a_1 z + a_2 z^2 + \cdots + a_n z^n + \cdots \]

be a function regular for \(|z| \leq 1\). The well known inequality of Fejér and Riesz asserts that

\[ \int_D |f(z)| \, |dz| \leq \frac{1}{2} \int_C |f(z)| \, |dz| , \]

where \(C\) is the circumference \(|z| = 1\), and \(D\) any of its diameters.¹

For \(f(z) = F'(z)\), the inequality (2) takes the form

\[ \int_D |F'(z)| \, |dz| \leq \frac{1}{2} \int_C |F'(z)| \, |dz| , \]

which shows that the total variation of \(F(z)\) along \(D\) does not exceed half of the total variation of \(F\) along \(C\). In this form the inequality remains valid for harmonic functions. Let \(z = \rho e^{i\theta}\). If \(U(z) = U(\rho, \theta)\) is harmonic for \(|z| \leq 1\), the total variation of \(F\) along \(D\) does not exceed half of the total variation of \(F\) along \(C\).²

In symbols,

\[ \int_D |U_\rho| \, d\rho \leq \frac{1}{2} \int_C |U_\theta| \, d\theta. \]

Let \(V(z) = V(\rho, \theta)\) be the harmonic function conjugate to \(U\). In (4) we may replace \(U_\rho\) by \(\rho^{-1} V_\theta\). Writing \(U_\theta = u, V_\theta = v\), we obtain an equivalent form of the inequality (4), namely

\[ \int_D \left| \frac{v(z)}{z} \right| \, |dz| \leq \frac{1}{2} \int_C |u(z)| \, |dz| . \]


It is valid for any pair of functions

\[ u(z) = u(\rho, \theta) = \frac{d_0}{2} + \sum_{n=1}^{\infty} (a_n \cos n\theta + b_n \sin n\theta)\rho^n, \]

\[ v(z) = v(\rho, \theta) = \sum_{n=1}^{\infty} (a_n \sin n\theta - b_n \cos n\theta)\rho^n, \]

harmonic and conjugate in \(|z| \leq 1\), provided that \(v(0) = 0\).

The purpose of this note is to prove the following complement to (4).

**Theorem 1.** Let

\[ U(z) = \frac{A_0}{2} + \sum_{n=1}^{\infty} (A_n \cos n\theta + B_n \sin n\theta)\rho^n \quad (z = \rho e^{i\theta}) \]

be a function harmonic for \(|z| \leq 1\), and let \(U_n(z)\) be the nth partial sum of the series (8). For every \(n\), the total variation of \(U_n(z)\) over \(D\) does not exceed \(2/\pi\) times the total variation of \(U(z)\) over \(C\). The constant \(2/\pi\) here cannot be replaced by any smaller number.

We shall prove this result in the following equivalent form.

**Theorem 1’.** Let \(u(z)\) and \(v(z)\), given by (6) and (7), be harmonic for \(|z| \leq 1\) and conjugate (in particular, \(v(0) = 0\)). Let \(v_n(z)\) be the nth partial sum of the series (7). Then

\[ \int_D \left| \frac{v_n(z)}{z} \right| \, |dz| \leq \frac{2}{\pi} \int_C |u(z)| \, |dz| \]

for all \(n\), the factor \(2/\pi\) on the right being the best possible.

Taking \(u(z) = zf(z)\), so that \(v(z) = -izf(z)\), we deduce from (9) the following corollary.

**Theorem 2.** Let \(f(z)\) be a function regular for \(|z| \leq 1\) and let \(s_n(z)\) be the nth partial sum of the series (1). Then for all \(n\)

\[ \int_D |s_n(z)| \, |dz| \leq \frac{2}{\pi} \int_C |f(z)| \, |dz|. \]

In all these results the assumption that the functions are harmonic (or regular) inside and on \(C\) can obviously be relaxed, and more general results may be obtained from the special ones by routine passages to the limit.\(^8\) Thus, for example, in Theorem 2 we may assume

\(^8\) See also §3, below.
that \( f(z) \) is regular for \(|z| < 1\) and continuous for \(|z| \leq 1\). The partial sums \( s_n(z)\) for such an \( f \) may be unbounded, and it is of interest to note that in the passage from (2) to (10) the increase of the coefficient on the right (from 1/2 to \(2/\pi = 0.64 \cdots\)) is not significant.

That \(2/\pi\) is the best constant in Theorem 1' (and so in Theorem 1) is easily seen from the example

\[
u(z) = 1/2 + \rho R \cos \theta + \rho^2 R^2 \cos 2\theta + \cdots \quad (z = \rho e^{i\theta})
\]

where \( R \) is a fixed positive number less than 1. The integral on the right of (9) is then \(\pi\). Taking \(n = 1\) and the segment \((-i, i)\) for \(D\) we find for the integral on the left the value \(2\pi R\). Since \( R \) may be as close to 1 as we wish, the conclusion follows.

As we shall see later, for each \(n > 1\) the constant \(2/\pi\) on the right can be replaced by a constant \(C_n < 2/\pi\), and clearly \(2/\pi = C_1 \geq C_2 \geq \cdots \). One might expect that \(C_n \to 1/2\), in accordance with (5). It is however not so, and the difference \(C_n - 1/2\) stays above a positive number. This fact has close connection with the Gibbs' phenomenon.

2. Proof of Theorem 1'. We shall use the formula

\[
v_n(\rho, \theta) = -\frac{1}{\pi} \int_{-\pi}^{\pi} u(1, \theta + t) \rho \Delta_n(\rho, t) dt,
\]

where

\[
\Delta_n(\rho, t) = \sum_{r=1}^{n} \rho^{r-1} \sin rt.
\]

Taking, as we may, for \(D\) the segment \((-1, +1)\) of the real axis, we get

\[
\int_D \left| \frac{v_n(z)}{z} \right| dz \leq \frac{1}{\pi} \int_{-\pi}^{\pi} \left| u(1, t) \right| \left\{ \int_{-1}^{1} \left| \Delta_n(\rho, t) \right| d\rho \right\} dt,
\]

and our problem reduces to showing that

\[
\int_{-1}^{+1} \left| \Delta_n(\rho, t) \right| d\rho \leq 2, \quad \text{for } n = 1, 2, \cdots, -\pi \leq t \leq \pi.
\]

That the integral on the left here is bounded as a function of \(n\) would be a simple matter to show. Complications arise when we want to show that the integral in question does not exceed 2.

We shall need the following lemma.
Lemma. Let $0 \leq t \leq \pi$. For every $\mu$ positive and odd, the values of the trigonometric polynomials

\[
\sin t/1 + \sin 3t/3 + \cdots + \sin \mu t/\mu, \\
\sin t/1 + \sin 3t/3 + \cdots + \sin (\mu - 2)t/(\mu - 2) + \sin \mu t/2\mu
\]

are contained between 0 and 1.

Taking temporarily this lemma for granted, we shall proceed with the proof of the inequality (13). Let us consider the auxiliary function

\[\Delta_n^*(\rho, t) = \sum_{r=1}^{n} \rho^{r-1} \sin \frac{rt}{t} = \sum_{r=1}^{n-1} \rho^{r-1} \sin \frac{rt}{t} + \frac{\rho^{n-1} \sin nt}{2}.\]

We easily find that

\[\Delta_n^*(\rho, t) = \frac{\sin t}{2} \left[ 2(1 - \rho^n \cos nt) - \rho^{n-1} \frac{\sin nt}{\sin t} (1 - \rho^2) \right].\]

We note that, for $0 \leq \rho \leq 1$,

\[2(1 - \rho^n \cos nt) - \rho^{n-1} \frac{\sin nt}{\sin t} (1 - \rho^2) \geq 2(1 - \rho^n) - np^{n-1}(1 - \rho^2) = (1 - \rho)[2(1 + \rho + \cdots + \rho^{n-1}) - \rho^{n-1}n(1 + \rho)] \geq (1 - \rho)(2np^{n-1} - 2np^{n-1}) = 0.\]

Thus $\Delta_n^*$ (unlike $\Delta_n$) is non-negative for $0 \leq t \leq \pi$, and so nonpositive in the interval $(-\pi, 0)$. This remark applies, in particular, to the trigonometric polynomial

\[\Delta_n^*(1, t) = \sum_{r=1}^{n} \sin \frac{rt}{t}.\]

Let us now fix $t$, $0 \leq t \leq \pi$. In the proof of (13) we shall consider various special cases.

Case (1). $\Delta_n(\rho, t) \geq 0$, $\Delta_n(\rho, t + \pi) \leq 0$ for $0 \leq \rho \leq 1$.

Thus

\[\int_{-1}^{1} |\Delta_n(\rho, t)| \, d\rho = \int_{0}^{1} [\Delta_n(\rho, t) - \Delta_n(\rho, t + \pi)] \, d\rho\]

\[= \int_{0}^{1} \left\{ \sum_{r \leq n} \rho^{r-1} \sin \frac{rt}{t} - \sum_{r \leq n} (-1)^r \rho^{r-1} \sin \frac{rt}{t} \right\} \, d\rho\]

\[= 2 \sum_{r \leq n, r \text{ odd}} \frac{\sin \frac{rt}{t}}{r} \leq 2,\]
by the lemma.

Case (ii). $\Delta_\alpha(\rho_0, t) < 0$, for some $\rho_0$, $0 \leq \rho_0 \leq 1$. We may assume that $0 < \rho_0 < 1$.

This cannot happen for $n = 1$. Thus $n \geq 2$. We shall show that in this case, for all $\rho$, $0 \leq \rho \leq 1$:

(a) $\Delta_\alpha(\rho, t + \pi) < 0$.

(b) $|\Delta_\alpha(\rho, t)|$ is majorized both by $\Delta_{n-1}(\rho, t)$ and $\Delta_{n+1}(\rho, t)$ (in particular, the latter quantities must be non-negative).

For suppose that

$$\Delta_\alpha(\rho, t) = \sin t + \rho \sin 2t + \cdots + \rho^{n-1} \sin nt < 0.$$ 

We know that for all $\rho$, $0 \leq \rho \leq 1$,

$$\Delta^\ast_\alpha(\rho, t) = \sin t + \rho \sin 2t + \cdots + \rho^{n-2} \sin (n-1)t + 2^{-1} \rho^{n-1} \sin nt \geq 0,$$

$$\Delta^\ast_{n+1}(\rho, t) = \sin t + \rho \sin 2t + \cdots + \rho^{n-1} \sin nt + 2^{-1} \rho^n \sin (n+1)t \geq 0.$$ 

A comparison of these inequalities shows that

$$\sin nt < 0, \quad \sin (n+1)t > 0.$$ 

Thus, since $\Delta_\alpha = \Delta^\ast_\alpha + 2^{-1} \rho^{n-1} \sin nt$, we have

$$2^{-1} \rho^{n-1} \sin nt \leq \Delta_\alpha(\rho, t) \leq \Delta^\ast_\alpha(\rho, t),$$

$$|\Delta_\alpha(\rho, t)| \leq |\Delta^\ast_\alpha(\rho, t)| + |2^{-1} \rho^{n-1} \sin nt|$$

$$= \Delta^\ast_\alpha(\rho, t) - 2^{-1} \rho^{n-1} \sin nt = \Delta_{n-1}(\rho, t).$$ 

Similarly, the formula $\Delta_\alpha = \Delta^\ast_{n+1} - 2^{-1} \rho^n \sin (n+1)t$ and (15) imply

$$-2^{-1} \rho^n \sin (n+1)t \leq \Delta_\alpha(\rho, t) \leq \Delta^\ast_{n+1}(\rho, t),$$

$$|\Delta_\alpha(\rho, t)| \leq |\Delta^\ast_{n+1}(\rho, t)| + |2^{-1} \rho^n \sin (n+1)t|$$

$$= \Delta^\ast_{n+1}(\rho, t) + 2^{-1} \rho^n \sin (n+1)t = \Delta_{n+1}(\rho, t).$$ 

Thus assertion (b) is proved. To prove (a) we replace $t$ by $t + \pi$ in the equations

$$\Delta_\alpha(\rho, t) = \Delta^\ast_\alpha(\rho, t) + 2^{-1} \rho^{n-1} \sin nt,$$

$$\Delta_\alpha(\rho, t) = \Delta^\ast_{n+1}(\rho, t) - 2^{-1} \rho^n \sin (n+1)t.$$ 

If $n$ is even, the first of these resulting equations, coupled with the inequalities $\Delta^\ast_\alpha(\rho, t + \pi) \leq 0, \sin nt < 0$, shows that $\Delta_\alpha(\rho, t + \pi) < 0$. If $n$ is odd, we similarly argue with the second equation. Thus (a) is proved.

Using (a) and (b), we see that in case (ii)
\[
\int_{-1}^{+1} | \Delta_n(p, t) | \, dp \leq \int_0^1 \{ \Delta_{n+1}(p, t) - \Delta_n(p, t + \pi) \} \, dp
\]
\[
= \int_0^1 \left\{ \sum_{r=1}^{n+1} r^{-1} \sin vt - \sum_{r=1}^n (-1)^r r^{-1} \sin vt \right\} \, dp.
\]

If \( n \) is even, we take the sign + in \( n \pm 1 \), and the last integral becomes
\[
2 \left\{ \sum_{r \leq n-1, r \text{ odd}} \frac{\sin vt}{r} + \frac{1}{2} \frac{\sin (n+1)t}{n+1} \right\} \leq 2,
\]
by the lemma. If \( n \) is odd \((n \geq 3)\), we take the sign − in \( n \pm 1 \), and the integral in question takes the form
\[
2 \left( \sum_{r \leq n-2, r \text{ odd}} \frac{\sin vt}{r} + \frac{1}{2} \frac{\sin nt}{n} \right) \leq 2,
\]
again by the lemma. Thus (13) holds in case (ii).

**Case (iii).** \( \Delta_n(p_0, t + \pi) > 0 \) for some \( p_0, 0 \leq p_0 \leq 1 \).

To prove that (13) holds in this case, we observe that, since \( \Delta_n(p, t) \) is odd in \( t \),
\[
\Delta_n(p_0, t + \pi) = - \Delta_n(p_0, \pi - t) = - \Delta_n(p_0, t'),
\]
where \( t' = \pi - t \). Thus \( 0 \leq t' \leq \pi, \Delta_n(p_0, t') \) is negative, and we are in case (ii). It follows that (13) holds if we replace there \( t \) by \( t' \). But
\[
\int_{-1}^{+1} | \Delta_n(p, t) | \, dp = \int_{-1}^{+1} | \Delta_n(p, t - \pi) | \, dp
\]
(18)
\[
= \int_{-1}^{+1} | \Delta_n(p, t') | \, dp \leq 2,
\]
and (13) holds again.

Cases (i), (ii), (iii) exhaust all possibilities, since the simultaneous occurrence of the inequalities \( \Delta_n(p_0, t) < 0, \Delta_n(p_1, t + \pi) > 0 \) is excluded by case (ii). Thus for the completion of the proof of Theorem 1' we need only prove the lemma.

In estimating the polynomial
\[
\sin t + \sin 3t + \cdots + \sin \mu t
\]
(19)
\[
= \int_0^t (\cos u + \cos 3u + \cdots + \cos \mu u) \, du
\]
\[
= \int_0^t \frac{\sin (\mu + 1)u}{2 \sin u} \, du,
\]
we may assume that $0 \leq t \leq \pi/2$, since replacing $t$ by $\pi - t$ does not change the value of the polynomial. It is clear that the maximum of the last integral is attained for $t=\pi/(\mu+1)$, and is equal to

$$
\int_0^{\pi/(\mu+1)} \frac{\sin (\mu + 1)u}{2 \sin u} \, du = \int_0^{\pi} \frac{\sin u}{2(\mu + 1) \sin (u/\mu + 1)} \, du.
$$

For fixed $u$, $0 \leq u \leq \pi$, and $\mu \geq 1$, the minimum of the last denominator is attained when $\mu = 1$. For this particular value of $\mu$ the last integral becomes $2^{-1} \int_0^\pi \cos(u/2) \, du = 1$, and one-half of the lemma is proved.

Similarly, assuming, as we may, that $\mu \geq 3$, we get

$$
\sin t + \frac{\sin 3t}{3} + \cdots + \frac{\sin \mu t}{2\mu} = \int_0^t \frac{\sin \mu u}{2 \tan u} \, du \leq \int_0^{\pi/\mu} \frac{\sin \mu u}{2 \sin u} \, du 
\leq \int_0^{\pi/\mu} \frac{\sin \mu u}{2 \sin u} \, du \leq 1
$$

by the result just obtained. This completes the proof of the lemma.  

3. Additional remarks. (i) Let $C_n$ be the least number such that

$$\int_D \left| \frac{v_n(z)}{z} \right| \, dz \leq C_n \int_D \left| u(z) \right| \, dz$$

for all functions $u(z)$ harmonic in $|z| \leq 1$. We know that $C_1 = 2/\pi$, and the argument just completed clearly shows that $C_n < 2/\pi$ for $n = 2, 3, \ldots$. The numbers $C_1 \geq C_2 \geq C_3 \geq \cdots$ tend to a limit $C^* \geq 1/2$. Combining the estimates in cases (i), (ii), (iii) with the inequalities of the lemma, we find that

$$C^* \leq \lim_{m \to \infty} \frac{1}{\pi} \int_0^{\pi/m} \frac{\sin mu}{\sin u} \, du = \lim_{m \to \infty} \frac{1}{\pi} \int_0^{\pi/m} \frac{\sin mu}{u} \, du = \frac{1}{\pi} \int_0^\pi \frac{\sin u}{u} \, du.
$$

On the other hand, let $R$ be any positive number less than 1 and let

$u(z) = u_R(z) = \pi^{-1}(1/2 + R \cos \theta + R^2 \rho^2 \cos 2\theta + \cdots), \quad z = \rho e^{i\theta},$

so that $\int_D \left| u(z) \right| \, dz = 1$. If $t$, $0 < t < \pi$, is the angle of $D$ with the positive real axis, then

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4 The author is indebted to a referee for pointing out a slip in the initial proof of the lemma.
\[
\int_D \left| \frac{v_n(z)}{z} \right| |dz| = \int_{-1}^{+1} \left| \Delta_n(\rho, t) \right| d\rho \\
\geq \int_{0}^{+1} \{ \Delta_n(\rho, t) - \Delta_n(\rho, t + \pi) \} d\rho \\
= 2 \left\{ R^\frac{\sin t}{1} + R^\frac{\sin 3t}{3} + \cdots + R^\frac{\sin \mu t}{\mu} \right\},
\]
where \( \mu \) is the largest odd integer not greater than \( n \). Taking \( R \) as close to 1 as we wish, we see that the last sum comes arbitrarily close to
\[
2 \left( \frac{\sin t}{1} + \frac{\sin 3t}{3} + \cdots + \frac{\sin \mu t}{\mu} \right),
\]
which is a partial sum of the Fourier series of the function \( (\pi/2) \) sign \( t \) \((-\pi < t < \pi)\). This function has a jump at \( t = 0 \) so that the partial sums (22) must display Gibbs' phenomenon. This also follows from (19) and (20), which show that
\[
C^* \geq \frac{1}{\pi} \int_0^\pi \frac{\sin u}{u} \, du.
\]
Comparing this with (21) we see that
\[
C^* = \frac{1}{\pi} \int_0^\pi \frac{\sin u}{u} \, du = .589490 \cdots.
\]
(ii) Let \( U(\rho, \theta) \) be the Poisson integral of a function \( F(t) \), \( 0 \leq t < 2\pi \), of bounded variation and not constant. Thus
\[
U(\rho, \theta) = \frac{1}{2\pi} \int_{-\pi}^{\pi} \frac{1 - \rho^2}{1 - 2\rho \cos(t - \theta) + \rho^2} F(t) \, dt.
\]
Since the total variation of \( U(z) \) on the circle \( |z| = \rho < 1 \) tends to the total variation of \( F(t) \) over \((0, 2\pi)\), Theorem 1 gives
\[
\int_D \left| \frac{d}{d\rho} U_n(\rho) \right| d\rho \leq \frac{2}{\pi} \int_0^{2\pi} \left| dF(t) \right|.
\]
This, of course, may be obtained directly, by applying to the formula
\[
\frac{d}{d\rho} U_n(\rho, \theta) = \frac{1}{\pi} \int_{-\pi}^{\pi} \Delta_n(\rho, \theta - t) \, dF(t)
\]
the estimates for \( \int_{-1}^{+1} \left| \Delta_n(\rho, t) \right| d\rho \). This direct approach shows that if,
for simplicity, we take for $D$ the segment $(-1, +1)$, then the sign of equality occurs in (23) if and only if (a) $n = 1$, (b) $F(t)$ is $Cx(t)$, where $C$ is any constant, and $x(t)$ equals $+1$ inside $(-\pi/2, +\pi/2)$ and equals $-1$ inside the intervals $(-\pi, -\pi/2)$ and $(\pi/2, \pi)$. In other words, $U(\rho, \theta)$ must be

$$C \sum_{r=1}^{\infty} (-1)^r \frac{\cos (2\nu - 1)\theta}{2\nu - 1} \rho^{2\nu - 1}.$$

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