A NOTE ON THE RIEMANN ZETA-FUNCTION

FU TRAING WANG

Let \( \rho_\alpha = \beta_\alpha + i\gamma_\alpha \) be the zeros of the Riemann zeta-function \( \zeta(1/2 + z) \) whose real part \( \beta_\alpha \geq 0 \). Then we have the following formula which is an improvement on Paley-Wiener's [1, p. 78]:

\[
\int_1^T \log \left| \frac{\zeta(1/2 + it)}{t^2} \right| dt = 2\pi \sum_{\alpha=1}^{\infty} \frac{\beta_\alpha}{|\rho_\alpha|^2} + \int_{x=2}^{T/2} R \{ e^{-i\theta} \log \zeta(1/2 + e^{i\theta}) \} d\theta + O\left( \frac{\log T}{T} \right).
\]

In order to prove this formula let \( \rho_\alpha \) \( (\nu = 1, 2, \ldots, n) \) be the zeros of \( \zeta(1/2 + z) \) for which \( 0 < \gamma_\alpha < T \) and \( 0 \leq \beta_\alpha < 1/2 \). We require the following lemma:

**Lemma.** Let \( K \) be the unit semicircle with center \( z = 0 \) lying in the right half-plane \( R(z) > 0 \) and let \( C \) be the broken line consisting of three segments \( L_1 \) \( (0 \leq x \leq T, \ y = T) \), \( L_2 \) \( (0 \leq x \leq T, \ y = -T) \) and \( L_3 \) \( (x = T, \ -T \leq y \leq T) \). Then

\[
\frac{1}{\pi} \int_1^T \log \left| \frac{\zeta(1/2 + it)}{t^2} \right| dt = 2 \sum_{\alpha=1}^{n} \frac{\beta_\alpha}{|\rho_\alpha|^2} + \frac{1}{2\pi i} \int_{K} \log \left( 1/2 + z \right) dz - \frac{1}{2\pi i} \int_{C} \log \left( 1/2 + z \right) dz.
\]

This is a form of Carleman's theorem which can be proved by a method of proof analogous to that of Littlewood's theorem (Titchmarsh [3, pp. 130–134]).

Let \( \Gamma \) be a contour describing \( C, K \) and the corresponding part of the imaginary axis, and let \( \rho_\alpha \) be a point interior to \( \Gamma \), and \( \log(z - \rho_\alpha) \) be taken as its principal value. We write \( G_1 \) as a contour describing \( \Gamma \) in positive direction to the point \( iy_\alpha \), then along the segment \( y = y_\alpha \), \( 0 < x < \beta_\alpha - r \), and describing a small circle with center \( z = \rho_\alpha \), radius \( r \), then going back along the negative side of this segment to \( iy_\alpha \), and then along \( \Gamma \) to the starting point.

By Cauchy's theorem we get

\[
\int_{C_1} \log \left( z - \rho_\alpha \right) \frac{dz}{z^2} = 0.
\]

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1 Numbers in brackets refer to the references cited at the end of the paper.
Hence
\[ \frac{1}{2\pi i} \int_{c} \frac{\log(z - \rho_{r})}{z^{2}} dz = -\int_{0}^{\beta_{r}} \frac{dx}{(x + iy)^{2}} \]
where the integral round the small circle with center \( z = \rho_{r} \), radius \( r \), tends to zero as \( r \to 0 \). This formula is also true for \( \beta_{r} = 0 \).

Put \( \xi(1/2 + z) = \phi(z) \prod_{n=1}^{\infty} (z - \rho_{n}) \prod_{n=1}^{\infty} (z - \bar{\rho}_{n}) \) where \( \phi(z) \) is regular and has no zero in and on \( T \). Then we get
\[
\frac{1}{2\pi i} \int_{c} \frac{\log \xi(1/2 + z)}{z^{2}} dz = \sum_{\gamma=1}^{n} \left( \frac{1}{\rho_{\gamma}} - \frac{1}{i\gamma} \right) + \sum_{\gamma=1}^{n} \left( \frac{1}{\bar{\rho}_{\gamma}} + \frac{1}{i\gamma} \right)
\]
\[
= 2 \sum_{\gamma=1}^{n} \frac{\beta_{\gamma}}{|\rho_{\gamma}|^{2}}.
\]

From this the lemma follows.

Now we have
\[ \log \xi(1/2 + x + iT) = O(1) \quad \text{for} \quad x \geq 1 \]
we have
\[ \int_{L_{1}} = \int_{0}^{1} \frac{\log \xi(1/2 + x + iT)}{(x + iT)^{2}} dx + O\left( \frac{1}{T} \right). \]

Since (Titchmarsh [2, p. 5])
\[ \arg \xi(1/2 + x + iT) = O(\log T) \quad \text{for} \quad 0 \leq x \leq 1 \]
and (Titchmarsh [2, p. 59])
\[ \log |\xi(1/2 + x + iT)| \]
\[ = \frac{1}{2} \sum_{|\gamma - T| < 1} \log \{(x - \beta)^{2} + (T - \gamma)^{2}\} + O(\log T), \]
then
\[ \int_{0}^{1} \frac{\log \xi(1/2 + x + iT)}{(x + iT)^{2}} dx = O\left( \frac{\log T}{T^{2}} \right). \]

From (3) and (4) we get
\[ \int_{L_{1}} = O\left( \frac{\log T}{T} \right). \]
Similarly

\[(6) \quad \int_{L_2} = O\left(\frac{\log T}{T}\right).\]

Since \(\log \zeta(1/2 + T + iy) = O(2^{-T})\), we get

\[(7) \quad \int_{L_2} = O(T2^{-T}).\]

By (1), (2), (5), (6) and (7) we have

\[(8) \quad \int_1^T \frac{\log |\zeta(1/2 + it)|}{t^2} \, dt = 2\pi \sum_{\gamma > T} \frac{\beta_\gamma}{|\gamma|^2}
+ \frac{1}{2i} \int_K \frac{\log \zeta(1/2 + z)}{z^2} \, dz + O\left(\frac{\log T}{T}\right).\]

But (Ingham [4, p. 70])

\[(9) \quad \sum_{\gamma > T} \frac{\beta_\gamma}{|\gamma|^2} = O\left(\sum_{\gamma > T} \frac{1}{\gamma^2}\right) = O\left(\frac{\log T}{T}\right).\]

The formula follows from (8) and (9).

Finally, if we make \(T \to \infty\) then

\[\int_1^\infty \frac{\log |\zeta(1/2 + it)|}{t^2} \, dt = \int_0^{\pi/2} R \{e^{-i\theta} \log \zeta(1/2 + e^{i\theta})\} \, d\theta\]
gives a necessary and sufficient condition for the truth of the Riemann hypothesis.

REFERENCES