where $C$ is an arbitrary analytic Jordan curve, $z=\alpha$ is a point interior to $C$, $f(z)$ is of class $E_\rho$ interior to $C$, and $n(z)$ is the modulus on $C$ of a function $N(z)$ analytic and nonvanishing in the closed region $\Gamma$, is

$$F_0(z) = A \left[ \frac{N(\alpha) \cdot g'(\alpha)}{N(z) \cdot g'(\alpha)} \right]^{1/\rho}.$$

Let $P_n(z)$ be the corresponding minimizing polynomial of degree $n$. Then the sequence $P_n(z)$, $n=0, 1, 2, \cdots$, converges maximally to $F_0(z)$ on $\Gamma$.

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NOTE ON THE LOCATION OF THE CRITICAL POINTS OF HARMONIC FUNCTIONS

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The object of this note is to publish the statement of the following theorem.

**Theorem I.** In the extended $(x, y)$-plane let $R_0$ be a simply-connected region bounded by a continuum $C_0$ not a single point, and let the disjoint continua $C_1, C_2, \cdots, C_n$ lie interior to $R_0$ and together with $C_0$ bound a subregion $R$ of $R_0$. By means of a conformal map of $R_0$ onto the unit circle we define in $R_0$ non-euclidean lines, the images of arbitrary circles orthogonal to the unit circle. Denote by $\Pi$ the smallest closed non-euclidean convex region in $R_0$ which contains $C_1, C_2, \cdots, C_n$.

Let the function $u(x, y)$ be harmonic interior to $R$, continuous in the closure of $R$, with the values zero on $C_0$ and unity on $C_1, C_2, \cdots, C_n$. Then the critical points of $u(x, y)$ in $R$ are $n-1$ in number and lie in $\Pi$.

Critical points are of course to be counted according to their multiplicities.

A limiting case of Theorem I has already been established: if $f(z)$

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is an analytic function whose modulus is constant on the boundary of a simply-connected region $R$, where $f(z)$ is analytic interior to $R$ and continuous in the closure of $R$, then the zeros of $f'(z)$ in $R$ lie in the smallest non-euclidean convex polygon in $R$ containing the zeros of $f(z)$ in $R$. Theorem I is readily established by the use of this limiting case, and of methods developed elsewhere by the present writer; details are left to the reader.

Theorem I admits an extension to the case where $R_0$ is bounded by $C_0$, and the subregion $R$ of $R_0$ is bounded by $C_0$ and by further disjoint continua $C_1$, $C_2$, $C_m$, $C_{m+1}$, $C_n$ in $R_0$; the function $u(x, y)$ is supposed harmonic interior to $R$, continuous in the closure of $R$, with the values zero on $C_0$, unity on $C_1$, $C_2$, $C_m$, and minus unity on $C_{m+1}$, $C_{m+2}$, $C_n$; a non-euclidean line $\Lambda$ in $R_0$ (if existent) which separates $C_1$, $C_2$, $C_m$ from $C_{m+1}$, $C_{m+2}$, $C_n$ cannot pass through a critical point of $u(x, y)$. If a $\Lambda$ exists, the points of $R_0$ which do not lie on any such $\Lambda$ form two disjoint non-euclidean convex point sets in $R_0$ which are closed with respect to $R_0$, which contain respectively $C_1$, $C_2$, $C_m$ and $C_{m+1}$, $C_{m+2}$, $C_n$, and which together contain all critical points of $u(x, y)$ in $R$. This extension of Theorem I may likewise be proved from a limiting case already formulated (loc. cit.) for a region $R_0$ bounded by a circle.

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