TOEPLITZ METHODS WHICH SUM A GIVEN SEQUENCE

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The following note arose out of discussions of a paper by Agnew, but is, however, self-contained.

**THEOREM.** Let \( \{x_n\} \) be a bounded divergent sequence. Suppose that \( \{y_n\} \) is summable by every regular Toeplitz method which sums \( \{x_n\} \). Then \( \{y_n\} \) is of the form \( \{cx_n + a_n\} \) where \( \{a_n\} \) is convergent.

**PROOF.** For typographical convenience we shall often write \( x(n) \) for \( x_n \), and so on. Let \( \{x(n_k)\}, k = 1, 2, \ldots, \) be any convergent subsequence of \( \{x_n\} \). Then \( \{x_n\} \) is summable by the matrix \( (a(n, k)) \) where \( a(n, k) = 1 \) for \( n = n_k \) and \( a(n, k) = 0 \) for \( n \neq n_k \). Hence \( \{y(n_k)\} \) is also convergent.

Let \( \{n'_k\} \) and \( \{n''_k\} \) be sequences of integers such that \( n'_k \neq n''_k \) for all \( k \) and

\[
\lim_{k \to \infty} x(n'_k) = A, \quad \lim_{k \to \infty} x(n''_k) = B, \quad A \neq B.
\]

These sequences \( \{n'_k\} \) and \( \{n''_k\} \) will be held fixed throughout the rest of the argument. Then the sequences \( \{y(n'_k)\} \) and \( \{y(n''_k)\} \) are also convergent, say to \( \alpha \) and \( \beta \) respectively. Let \( \{x(n_k)\} \) be an arbitrary convergent subsequence of \( \{x_n\} \) with the limit \( C \). Let \( \lambda \) and \( \mu \) be determined by the equations

\[
\lambda + \mu = 1, \quad \lambda A + \mu B = C.
\]

Then the matrix \( (b(n, k)) \) with

\[
b(n, k) = \begin{cases}
\lambda, & n = n'_k, \quad k \text{ even}, \\
\mu, & n = n''_k, \quad k \text{ even}, \\
1, & n = n_k, \quad k \text{ odd}, \\
0, & \text{for all other values of } n \text{ and } k,
\end{cases}
\]

sums \( \{x_n\} \) to the limit \( C \). Hence it also sums \( \{y_n\} \), that is

\[
\lim_{k \to \infty} y(n_k) = \lim_{k \to \infty} (\lambda y(n'_k) + \mu y(n''_k)) = \lambda \alpha + \mu \beta.
\]

Note that the numbers \( \lambda \) and \( \mu \) depend only on \( C \) and not on the particular subsequence \( \{x(n_k)\} \) converging to \( C \), and hence \( \lim_{k \to \infty} y(n_k) \)

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also depends only on $C$, and is, in fact, a linear function of $C$.

We now determine constants $m$ and $a$ from the equations

$$\alpha = mA + a, \quad \beta = mB + a.$$  

Let $\{n_k\}$ be an arbitrary sequence of positive integers, and let $\{n_k''\}$
be a subsequence of $\{n_k\}$ such that $\{x(n_k''')\}$ converges, say to $C$.
We determine $\lambda$ and $\mu$ as before. Then

$$\lim_{k \to n} (y(n_k') - mx(n_k'')) = \lambda \alpha + \mu \beta - mC$$
$$= \lambda(mA + a) + \mu(mB + a) - mC = a.$$  

Thus every subsequence of $\{y_n - mx_n\}$ contains a subsubsequence converging to $a$. Hence $\lim_{n \to \infty} (y_n - mx_n) = a$, which proves our theorem.

**Corollary.** If $\{x_n\}$ and $\{y_n\}$ are bounded divergent sequences, and
$\{y_n\}$ is summable by every regular Toeplitz method which sums $\{x_n\}$,
then $\{x_n\}$ is summable by every regular Toeplitz method which sums $\{y_n\}$.

By a theorem of Agnew, there is no single Toeplitz method which has the sequences of the form $\{cx_n + a_n\}$ as its convergence field. The
above theorem shows, however, that this set of sequences is the common part of the convergence fields of Toeplitz methods which sum $\{x_n\}$.

Added November 11, 1945 We have just had the opportunity of
seeing the paper of A. Brudno, *Summation of bounded sequences by
matrices*, Rec. Math. (Mat. Sbornik) N.S. vol. 16 (1945), pp. 191–247. From the English summary it seems that our result is con­tained in his Theorem 11, p. 236. His Theorem 11 can clearly be proved by our method. It is difficult to compare the simplicity of the proofs.

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