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ON THE SUMMATION OF MULTIPLE FOURIER SERIES. III

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Let \( f(x) = f(x_1, \ldots, x_k) \) be a function of the Lebesgue class \( L \), which is periodic in each of the \( k \)-variables, having the period \( 2\pi \). Let

\[
a_{n_1 \ldots n_k} = \frac{1}{(2\pi)^k} \cdot \int_{-\pi}^{+\pi} \cdots \int_{-\pi}^{+\pi} f(x) \exp \left\{ -i(n_1 x_1 + \cdots + n_k x_k) \right\} dx_1 \cdots dx_k,
\]

where \( \{n_j\} \) are all integers. Then the series \( \sum a_{n_1 \ldots n_k} \exp i(n_1 x_1 + \cdots + n_k x_k) \) is called the multiple Fourier series of the function \( f(x) \), and we write

\[
f(x) \sim \sum a_{n_1 \ldots n_k} \exp i(n_1 x_1 + \cdots + n_k x_k).
\]

Let the numbers \( \nu_1^2 + \cdots + \nu_k^2 \), when arranged in increasing order of magnitude, be denoted by \( \lambda_0 < \lambda_1 < \cdots < \lambda_n < \cdots \), and let

\[
C_n(x) = \sum a_{n_1 \ldots n_k} \exp i(n_1 x_1 + \cdots + n_k x_k),
\]

where the sum is taken over all \( \nu_1^2 + \cdots + \nu_k^2 = \lambda_n \),

\[
\phi(x, t) = \sum C_n(x) \exp (-\lambda_n t),
\]

\[
S_R(x) = \sum_{\lambda_n \leq R^2} C_n(x), \quad \lambda_n \leq R^2 < \lambda_{n+1}.
\]

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1 Papers I and II with the same title are to appear in Proc. London Math. Soc.
Also, let \( R_k(\lambda) \) and \( r_k(\lambda) \) represent respectively the number of solutions of \( v_1^2 + \cdots + v_k^2 \leq \lambda \) and of \( v_1^2 + \cdots + v_k^2 = \lambda \).

The object of this note is to study the convergence of multiple Fourier series, when summed up spherically by Bochner's method, that is, of the series \( \sum C_n(x) \). We prove the following results.

**Theorem I.** If

\[
\sum (v_1^2 + \cdots + v_k^2)^{k/2} |a_{v_1 \cdots v_k}|^2 < \infty,
\]

then the series \( \sum_{n=0}^{\infty} C_n \) converges at every point of continuity of \( f(x) \).

**Theorem II.** If

\[
\sum (v_1^2 + \cdots + v_k^2)^{(k/2)+\epsilon} |a_{v_1 \cdots v_k}|^2 < \infty, \quad \epsilon > 0,
\]

then the series \( \sum_{n=0}^{\infty} C_n \) converges absolutely.

The following result of Bochner is used in the proof of the above theorems.

**Lemma.** At a point of continuity of \( f(x) \), \( \phi(x, t) \) tends to a limit as \( t \) tends to zero.

**Proof of Theorem I.** We shall first prove that

\[
\lim_{R \to \infty} S_R(x) = \lim_{t \to +0} \phi(x, t),
\]

whenever the limit on the right exists. Next, by the application of the above lemma, we deduce that at a point where \( f(x) \) is continuous, \( \sum C_n(x) \) is convergent.

Now

\[
S_R(x) - \phi(x, t) = \sum_{s=0}^{n} C_s [1 - \exp (-\lambda s t)] - \sum_{s=n+1}^{\infty} C_s \exp (-\lambda s t)
\]

(2)

\[
= \sum J_1 - J_2,
\]

say. We have,

\[
J_1 = \sum_{s=0}^{n} C_s [1 - \exp (-\lambda s t)]
\]

\[
= \sum_{s=0}^{n} [1 - \exp (-\lambda s t)] \sum a_{s_1 \cdots s_k} \exp \left[ i(v_1 x_1 + \cdots + v_k x_k) \right]
\]

\[
= \sum a_{s_1 \cdots s_k} \exp \left[ i(v_1 x_1 + \cdots + v_k x_k) [1 - \exp (-v_1^2 - \cdots - v_k^2) t] \right],
\]

where the third sum runs over \( \lambda_s = \lambda_1^2 + \cdots + \lambda_k^2 \) and the last sum runs over \( \lambda_1^2 + \cdots + \lambda_k^2 \leq \lambda_n \), so that

\[
|J_1| \leq \sum_{a_1, \ldots, a_k} \left| 1 - \exp \left( - \frac{\lambda^2}{2} - \cdots - \frac{\lambda_k^2}{2} \right) \right|
\leq \left| \sum_{a_1, \ldots, a_k} \left( \frac{\lambda_1^2}{2} + \cdots + \frac{\lambda_k^2}{2} \right) \right| \left| a_1, \ldots, a_k \right|^2
\leq O(1) \cdot t \left[ \sum_{s=0}^{n} r_k(\lambda_s) \lambda_s^{2-k/2} \right]^{1/2},
\]

where the first sum runs over \( \lambda_1^2 + \cdots + \lambda_k^2 \leq \lambda_n \).

Now,

\[
\sum_{s=0}^{n} r_k(\lambda_s) \lambda_s^{2-k/2} = \sum_{s=0}^{n-1} R_k(\lambda_s) \left\{ \lambda_s^{2-k/2} - \lambda_{s+1}^{2-k/2} \right\} + R_k(\lambda_n) \lambda_n^{2-k/2}
= O \left( \int_{0}^{\lambda_n} x d x \right) + O(\lambda_n^2)
= O(\lambda_n^2).
\]

Hence, from (3), we obtain,

\[
|J_1| = O(\lambda_n).
\]

Again,

\[
|J_2| = \left\| \sum_{s=n+1}^{\infty} C_s \exp \left( - \lambda_s t \right) \right\|
\leq \lambda_n^{-k/4} \sum_{s=n+1}^{\infty} \lambda_s^{k/4} \left| C_s \exp \left( - \lambda_s t \right) \right|
\leq \lambda_n^{-k/4} \left[ \sum_{a_1, \ldots, a_k} \left( \frac{\lambda_1^2}{2} + \cdots + \frac{\lambda_k^2}{2} \right) \left| a_1, \ldots, a_k \right|^2 \right]^{1/2}
\leq \varepsilon_n \left( \lambda_n \right)^{-k/4}
\]

(in the last two sums \( \lambda_1^2 + \cdots + \lambda_k^2 \) runs from \( \lambda_n+1 \) to \( \infty \)), where

\[
\sum_{s=n+1}^{\infty} \left( \frac{\lambda_1^2}{2} + \cdots + \frac{\lambda_k^2}{2} \right) \left| a_1, \ldots, a_k \right|^2 = \varepsilon_n
\]

(\( \lambda_1^2 + \cdots + \lambda_k^2 \) runs from \( \lambda_n+1 \) to \( \infty \), and \( \varepsilon_n \to 0 \) as \( n \to \infty \)), since

\[
\sum \exp \left\{ -2(\lambda_1^2 + \cdots + \lambda_k^2) t \right\} = O(\varepsilon^{-1/2}) \quad \text{as} \quad t \to 0.
\]

Thus, we have, from (5) and (6),
\[
S_R(x) - \phi(x, t) = O(\delta_n) + O(\epsilon_n^{1/2} \delta_n^{-4/k}).
\]

If \( t \) is so chosen that \( t\lambda_n = \delta_n = \epsilon_n^{1/2} \), then,

\[
S_R(x) - \phi(x, t) = O(\delta_n) + O(\epsilon_n^{1/2} \delta_n^{-4/k}) = o(1), \quad \text{as } n \to \infty.
\]

**Proof of Theorem II.**

\[
\sum |C_n(x)| \leq \sum |a_n \cdots a_k| \\
\leq \left\{ \sum (v_1^2 + \cdots + v_k^2)^{1/2} |a_n \cdots a_k|^{2/2} \right\} \\
\times \left\{ \sum (v_1^2 + \cdots + v_k^2)^{-1/2} \right\}^{1/2} \\
= O(1) \sum (r_k(\lambda_n)^{-1/2} \lambda_n^{-1/2 - \epsilon})^{1/2} \\
= O\left( \left( \int_\mathbb{R} R_k(x) x^{-k/2 - 1 - \epsilon} dx \right)^{1/2} \right) \\
= O\left( \left( \int_\mathbb{R} x^{-1-\epsilon} dx \right)^{1/2} \right) < \infty.
\]

On using Hölder’s inequality instead of Schwarz’s in (3) and (6), we can easily generalize Theorem I as follows:

*If*

\[
\sum (v_1^2 + \cdots + v_k^2)^{k(p-1)/2} |a_n \cdots a_k| < \infty,
\]

*where* \( 1 < p \leq 2 \), *then* \( \sum C_n \) *converges at every point of continuity of* \( f(x) \).

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