SOME REMARKS ABOUT ADDITIVE AND MULTIPLICATIVE FUNCTIONS

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The present paper contains some results about the classical multiplicative functions \( \phi(n) \), \( \sigma(n) \) and also about general additive and multiplicative functions.

(1) It is well known that \( n/\phi(n) \) and \( \sigma(n)/n \) have a distribution function.\(^1\) Denote these functions by \( f_1(x) \) and \( f_2(x) \). \( f_1(x) \) denotes the density of integers for which \( n/\phi(n) \leq x \). It is known that both \( f_1(x) \) and \( f_2(x) \) are strictly increasing and purely singular.\(^1\) We propose to investigate \( f_1(x) \) and \( f_2(x) \); we shall give details only in case of \( f_1(x) \).

First we prove the following theorem.

**Theorem 1.** We have for every \( \epsilon > 0 \) and sufficiently large \( x \)

\[
\exp \left( - \exp \left[ (1 + \epsilon)ax \right] \right) < 1 - f_1(x) < \exp \left( - \exp \left[ (1 - \epsilon)ax \right] \right)
\]

where \( a = \exp(-\gamma) \), \( \gamma \) Euler's constant.

We shall prove a stronger result. Put \( A_r = \prod_{i=1}^{r} p_i \), \( p_i \) consecutive primes. Define \( A_k \) by \( A_k/\phi(A_k) \geq x > A_{k-1}/\phi(A_{k-1}) \). Then we have

\[
1/A_k < 1 - f_1(x) < 1/A_k^{1-\epsilon}.
\]

First of all it is easy to see that Theorem 1 follows from (2), since from the prime number theorem we easily obtain that \( \log \log A_k = (1+o(1))ax \), which shows that (1) follows from (2).

(2) means that the density of integers with \( \phi(n) \leq (1/x)n \) is between \( 1/A_k \) and \( 1/A_k^{1-\epsilon} \).

We evidently have for every \( n \equiv 0 \pmod{A_k} \), \( n/\phi(n) \geq x \), which proves

\[
1/A_k \leq 1 - f_1(x).
\]

To get rid of the equality sign, it will be sufficient to observe that there exist integers \( u \) with \( u/\phi(u) \geq x \), \( (u, A_k) = 1 \), and that the density of the integers \( n \equiv 0 \pmod{u} \), \( n \not\equiv 0 \pmod{A_k} \) is positive. This proves the first part of (2). The proof of the second part will be much harder. We split the integers satisfying \( n/\phi(n) \geq x \) into two classes. In the first class are the integers which have more than \( \left[ (1 - \epsilon_1)k \right] = r \) prime factors not greater than \( Bp_k \), where \( B = B(\epsilon_1) \) is a large number. In

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\(^1\) These results are due to Schönberg and Davenport. For a more general result see P. Erdős, J. London Math. Soc. vol. 13 (1938) pp. 119–127.
the second class are the other integers satisfying \( n/\phi(n) \geq x \). It is easy to see that the number of integers of the first class does not exceed

\[
2^{\pi(B'p_k)}/A_r = 2^\phi(p_k)/A_r < 1/A_k^{1-\epsilon}
\]

since \( \pi(B'p_k) = o(p_k) \) (\( \pi(x) \) denotes the number of primes not greater than \( x \)), and from the prime number theorem \( \log A_r > (1-\epsilon)p_k \) if \( \epsilon_1 \) is small.

Let now \( n \) be any integer of the second class. A simple argument shows that

\[
\prod_{p|n} \left(1 - \frac{1}{p}\right) < \prod_{t=1}^{k-1} \left(1 - \frac{1}{p_t}\right) < 1 - \frac{c_1\epsilon_1}{\log p_k}.
\]

The prime indicates that the product is extended over the \( p > B'p_k \). The first inequality follows from the definition of \( A_k \), and from the fact that \( n \) is of the second class, the second inequality follows from the prime number theorem. Thus we have

\[
\sum_{p|n} \frac{1}{p} > \frac{c_1\epsilon_1}{\log p_k}.
\]

Denote now by \( J_t \) the interval \( (B'tp_k, B^{t+1}p_k) \), \( t = 1, 2, \cdots \). It follows from (4) that for every integer of the second class there exists some \( t \) such that

\[
\sum_{p|n} \frac{1}{p} > c_1\epsilon_1 / 2^t \log p_k
\]

where in \( \sum_t \) the summation is extended over the primes in \( J_t \). Thus for some \( t, n \) must divide more than

\[
c_1\epsilon_1(B^t/2^t)(p_k/\log p_k) = B_t
\]

primes in \( J_t \). The density of the integers satisfying (6), that is, the density of the integers of the second class, is less than

\[
\sum_{t=1}^{\infty} \left( \sum_{p \in J_t} \frac{1}{p} \right) B_t! / [B_t]! < \frac{1}{[B_t]!} < e^{-2p_k} < \frac{1}{A_k},
\]

that is, \( \sum_{p \in J_t} 1/p < 1 \) for large enough \( k \) (\( B \) is independent of \( k \), if \( B = B(\epsilon_1) \) is large enough. Theorem 1 now follows from (3) and (7).

From Theorem 1 we easily obtain that

\[
\lim_{x \to \infty} \frac{1}{x} \sum_{n=1}^{x} \exp(\phi(n))
\]

exists. In fact we can also prove that for \( \alpha < \alpha \)
$$\lim_{n \to \infty} \frac{1}{x} \sum_{n=1}^{x} \exp \left( \exp \left( \phi(n) \right) \right)$$

exists. For $\alpha > a$ the limit is infinite.

**Theorem 2.**

$$\frac{1}{A_k^{1+\varepsilon}} < 1 - f_1(x) < \frac{1}{A_k^{1-\varepsilon}}.$$  

We omit the proof since it is very similar to that of Theorem 1.

**Theorem 3.** Let $\varepsilon \to 0$, then

$$f_1(1 + \varepsilon) = (1 + o(1))a/\log \varepsilon^{-1}, \quad f_2(1 + \varepsilon) = (1 + o(1))a/\log \varepsilon^{-1}.$$  

We prove only the first statement since the proof of the second is essentially the same. Let $n$ be an integer with $n/\phi(n) \leq 1 + \varepsilon$. Clearly $n$ does not divide any prime $p < (1 - (1 + \varepsilon)^{-1})^{-1} = e^{-1} + O(1)$. Thus

$$f_1(1 + \varepsilon) < (1 + o(1))a/\log \varepsilon^{-1}. \tag{8}$$

Denote by $J_t$ the interval

$$(4t^{-1}(1 - (1 + \varepsilon)^{-1})^{-1}, 4t(1 - (1 + \varepsilon)^{-1})^{-1}).$$

If an integer $n \equiv 0 \pmod{p_t}$, $p_t < (1 - (1 + \varepsilon)^{-1})^{-1}$, does not satisfy $n/\phi(n) \leq 1 + \varepsilon$, then a simple computation shows that for some $t$ it must have at least $t$ prime factors in $J_t$. Thus the number of these integers does not exceed

$$(1 + o(1)) \frac{a}{\log \varepsilon^{-1}} \sum_{t=1}^{\infty} \left( \sum_{p \in J_t} \frac{1}{p} \right)^t / t! = o(a/\log \varepsilon^{-1}),$$

which together with (8) proves Theorem 3.

It follows from Theorem 3 that $f_1'(1) = \infty$. It would be easy to show that $f_1'(n/\phi(n)) = \infty$ for every $n$.

Denote by $f_1^\alpha$ and $f_2^\alpha$ the distribution functions of

$$\prod_{p|n} \left( 1 - \frac{1}{p} \right)^{-\alpha} \quad \text{and} \quad \sum_{d|n} \frac{1}{d^\alpha}, \quad \alpha > 0.$$  

**Theorem 4.**

$$f_1^{(\alpha)}(1 + \varepsilon) = (1 + o(1)) \frac{a\alpha}{\log \varepsilon^{-1}}, \quad f_2^{(\alpha)} = (1 + o(1)) \frac{a\alpha}{\log \varepsilon^{-1}}.$$  

We omit the proof since it is very similar to that of Theorem 3.

Let us denote by $F_\alpha(x)$, $\alpha > 0$, the distribution function of $\prod_{p|n}(1 - \log p^x)^{-1}, \alpha > 0$. 
Theorem 5.

\[ F_1(1 + \varepsilon) = (1 + o(1))b\varepsilon, \]

that is, \( F'_1(1) = b \). Also \( F'_\alpha(1) = 0 \) for \( \alpha < 1 \) and \( F'_\alpha(1) = \infty \) for \( \alpha > 1 \).

We do not give the details of the proof since it would be long and similar to that of Theorem 3. We just make the following remarks:

If \( n \) satisfies

\[ \sum_{p|n} \frac{1}{\log p} \leq 1 + \varepsilon \]

then \( n \) does not divide any prime \( p \leq \exp(1/\varepsilon) \). Thus \( F'_1(1 + \varepsilon) \leq (1 + o(1))a\varepsilon \). But here (unlike in Theorem 3) we have \( F'_1(1 + \varepsilon) = (1 + o(1))b, \ b < a \). We obtain analogous results if we consider the additive function \( \sum_{p|n} 1/\log p \). It is possible that \( F'_1(x) \) exists for every \( 1 \leq x \), but this we can not prove.

(2) The following results are well known:

\[ \sum_{m=1}^{x} \frac{\phi(m)}{m} = (1 + o(1)) \frac{6}{\pi^2} x, \quad \sum_{m=1}^{x} \frac{\sigma(m)}{m} = (1 + o(1)) \frac{\pi^2}{6} x. \]

The density of integers for which \( \sigma(n+1)/(n+1) > \sigma(n)/n \) is 1/2, also the density of integers for which \( \phi(n+1)/(n+1) > \phi(n)/n \) is 1/2.²

Now we prove the following theorem.

Theorem 6. Let \( g(n)/\log \log \log n \to \infty \). Then we have

(i)

\[ \sum_{m=n}^{n+g(n)} \frac{\phi(m)}{m} = (1 + o(1)) \frac{6}{\pi^2} g(n). \]

(ii) The number of integers \( m \) in \( (n, n+g(n)) \) which satisfy \( \phi(m+1)/(m+1) > \phi(m)/m \) equals \( (1 + o(1))g(n)/2 \).

(iii) The number of integers \( m \) in \( (n, n+g(n)) \) which satisfy \( m/\phi(m) \leq c \) equals \( (1 + o(1))g(n)f_1(c) \). In other words the distribution function of \( \phi(m)/m \) in \( (n, n+g(n)) \) is the same as the distribution function of \( \phi(m)/m \).

All these results are best possible; they become false if for infinitely many \( n, g(n) < c \log \log \log n \).

We prove only (i); the proof of (ii) and (iii) are similar. Let \( A = A(n) \) tend to infinity sufficiently slowly. Put

\[ \frac{\phi(m)}{m} = D_1(m)D_2(m), \]

where
\[ D_1(m) = \prod_{p|m} \left( 1 - \frac{1}{p} \right), \quad D_2(m) = \prod_{p|m} \left( 1 - \frac{1}{p^2} \right). \]

The prime indicates that \( p \leq A \), the two primes that \( p > A \). We evidently have
\[
\sum_{m=n}^{n+g(n)} \frac{\phi(m)}{m} < \sum_{m=n}^{n+g(n)} D_1(m) = \sum_{d} \left( \frac{g(n)}{d} \right) \frac{\mu(d)}{d} = (1 + o(1)) g(n) \prod_{p \leq A} \left( 1 - \frac{1}{p} \right) = (1 + o(1)) \frac{\pi^2}{6} g(n)
\]
where the three primes indicate that the prime factors of \( d \) are not greater than \( A \), and \( (g(n)/d) \) denotes the number of multiples of \( d \) in \((n, n+g(n))\). Now we show that for sufficiently large \( A \) the number of integers in \((n, n+g(n))\) which satisfy
\[
D_2(m) < 1 - \epsilon
\]
is \( o(g(n)) \). It will be sufficient to show that
\[
\prod_{m} D_2(m) > (1 - \eta)^{\sigma(n)}
\]
for every \( \eta > 0 \), the product over \( m \) runs in \((n, n+g(n))\). We evidently have
\[
\prod_{m} D_2(m) > \prod_{1} \left( 1 - \frac{1}{p^{\sigma(n)/p-1}} \right) \prod_{2} \left( 1 - \frac{1}{p^2} \right)
\]
where, in \( \prod_{1} \), \( A < p \leq g(n) \), and in \( \prod_{2} \), \( p \) runs through the prime factors greater than \( g(n) \) of \( n(n+1) \cdots (n+g(n)) \). Clearly
\[
\prod_{1} > \prod_{p > A} \left( 1 - \frac{\epsilon}{p^2} \right)^{\sigma(n)} > (1 - \eta_1)^{\sigma(n)}.
\]

From the prime number theorem we have \( \prod_{p \leq x} p < e^{2x} \). Thus
\[
\prod_{2} > \prod_{p \leq 2y} \left( 1 - \frac{1}{p} \right) > \frac{c_1}{\log \frac{y}{2}}
\]
where \( y = \log [n(n+1) \cdots (n+g(n))] \). Hence using \( g(n)/\log \log \log n \rightarrow \infty \), we obtain by a simple calculation that
\[
\prod_{2} > (1 - \eta_2)^{\sigma(n)}
\]
which proves (11) and therefore (10). From (9) and (10) we obtain by a simple argument that
Now we are going to prove that (i) is best possible. Put \( g(N) = c \log \log \log N \), \( n/2 < N < n \). Further let \( A_1, A_2, \ldots, A_r \), \( r = \lceil 2^{-1} \log \log \log n \rceil \) be relatively prime integers all of whose prime factors are less than \( 2^{-1} \log \log n \) and for which

\[
\frac{1}{4} < \frac{\varphi(A_i)}{A_i} < \frac{1}{2}, \quad i = 1, 2, \ldots, r.
\]

This is obviously possible since

\[
\prod_{p<\lceil \log \log n \rceil/2} \left( 1 - \frac{1}{p} \right) < \frac{c}{\log \log n} < \left( \frac{1}{4} \right)^{\log \log \log n}/2.
\]

Now choose \( n/2 < N < n \) so that \( N+j \equiv 0 \pmod{A_i}, j \leq r \). This is possible since by the prime number theorem \( A_1 \cdot A_2 \cdot \cdots \cdot A_r < n/2 \). (In all cases where we refer to the prime number theorem a more elementary result would be sufficient.) Clearly

\[
\sum_{m=N+1}^{N+\varphi(N)/2} \frac{\phi(m)}{m} \leq \frac{\log \log \log n}{4}.
\]

From (9) we have

\[
\sum_{m=N+1}^{N+\varphi(N)/2} \frac{\phi(m)}{m} < (1 + o(1)) \frac{6}{\pi^2} \left( g(N) - \frac{\log \log \log n}{2} \right).
\]

Thus finally from (10) and (11) we obtain by a simple calculation

\[
\sum_{m=N}^{N+\varphi(N)/2} \frac{\phi(m)}{m} < (1 - c) \frac{6}{\pi^2} g(N),
\]

which shows that (i) is best possible.\(^4\)

**Theorem 7.** Let \( g_1(n)/\log \log n \to \infty \). Then we have

(i) \[
\sum_{m=n}^{n+g_1(n)} \frac{\sigma(m)}{m} = (1 + o(1)) \frac{\pi^2}{6} g_1(n).
\]

(ii) Let \( g_2(n)/\log \log \log n \to \infty \). The number of integers \( m \) in \((n, n+g_2(n))\) which satisfy \( \sigma(n+1)/(n+1) > \sigma(n)/n \) equals \((1+o(1)) \cdot g(n)/2\).

\(^8\) This proof is similar to a proof in P. Erdős, J. London Math. Soc. vol. 10 (1935) pp. 128–131.

\(^4\) This proof is similar to a proof of Chowla and Pillai, J. London Math. Soc. vol. 5 (1930) pp. 95–101.
(iii) The number of integers \( m \) in \( (n, n+g(n)) \) which satisfy \( \sigma(m)/m < c \) equals \( (1+o(1)) \) \( g(n) f_\delta(c) \). All these results are best possible.

We omit the proof of Theorem 7, since it is similar to that of Theorem 6. We must allow \( g_1(n)/\log \log n \to \infty \), since it is well known that for some \( m \leq n \), \( \sigma(m) > c \log \log n \) (for example, \( m = \prod_{p < (\log n)/2} p \)).

Let \( f(n) \leq 1 \) and \( F(n) \geq 1 \) be multiplicative functions with

\[
\sum_p \frac{1 - f(p)}{p} < \infty \quad \text{and} \quad \sum_p \frac{F(p) - 1}{p} < \infty.
\]

Then we have:

**Theorem 8.** Let \( A = A(n) \) tend to infinity arbitrarily slowly, then

\[
\frac{1}{A} \sum_{m=n}^{n+A} f(m) < (1 + o(1)) \frac{1}{n} \sum_{m=1}^{n} f(m)
\]

and

\[
\frac{1}{A} \sum_{m=n}^{n+A} F(m) > (1 + o(1)) \frac{1}{n} \sum_{m=1}^{n} F(m).
\]

The proof is quite trivial; it is similar to that of (9). It can be shown that \( \lim (1/n) \sum_{m=1}^{n} f(m) \) and \( \lim (1/n) \sum_{m=1}^{n} F(m) \) exist.

Denote by \( V(n) \) the number of prime factors of \( n \) and by \( d(n) \) the number of divisors of \( n \). We can prove analogs to Theorem 6 for these functions. But the results are very unsatisfactory since for \( v(n) \) we have to choose \( g(n) = n^{v/\log \log n} \) and for \( d(n) \), \( g(n) = n^c \) for some suitable \( c \). These results are probably very far from best possible.

(3) Let \( n = p_1^{a_1} p_2^{a_2} \cdots p_k^{a_k} \), \( p_1^{a_1} < p_2^{a_2} < \cdots < p_k^{a_k} \). Put \( (p_i^{a_i})^{b_i} = p_i^{a_i+1} \). We prove the following theorem.

**Theorem 9.** Let \( 1 < x \), then for almost all \( n \) the number of \( b \)'s greater than \( x \) equals \( x^{-1} \log \log n + o(\log \log n) \).

**Remark.** We immediately obtain that every interval \( (x, x+\epsilon) \) contains \( (1+o(1)) (\epsilon/x(x+\epsilon)) \log \log n \) \( b \)'s.

We are going to give only an outline of the proof. First of all we can assume that all the \( \alpha \)'s are 1, since for large \( r \) the number of integers not greater than \( n \) for which \( r \) or more of the \( \alpha \)'s is greater than 1 is less than \( en \), since the number of these integers is clearly less than

\[
\left( \sum_p \frac{1}{p^r} \right)^r / r! < en.
\]

* This result has been stated previously, see footnote 4.
Denote by \( F(n) \) the number of prime factors \( p \) of \( n \) such that no prime \( q \) in \( (p, p^2) \) divides \( n \). \( F(n) \) is thus the number of \( b \)'s not less than \( x \). We have

\[
F(n) \sim \frac{1}{x} \log \log n + o(\log \log n).
\]

We now give a sketch of the proof. Clearly

\[
\sum_{m=1}^{n} F(m) = \sum_{p} f_p(n)
\]

where \( f_p(n) \) denotes the number of integers \( m \leq n \), with \( m \equiv 0 \mod{p} \) and \( m \not\equiv 0 \mod{q} \), \( p < q < p^2 \). It is easy to see that for \( p < n^* \)

\[
f_p(n) = (1 + o(1))n/p^x \quad (p \text{ large}).
\]

Also for all \( p \)

\[
f_p(n) \leq n/p.
\]

Thus

\[
\sum_{m=1}^{n} F(m) = \sum_{x \leq n^*} \frac{n}{px} + O \sum_{n^* < p < n} \frac{n}{p} + o(\log \log n)
\]

\[
= (1 + o(1)) \frac{\log \log n}{x},
\]

which proves (14). Now we have to show that

\[
F(m) = (1 + o(1))(\log \log n)/x
\]

for almost all \( m \leq n \). We use Turán's method.\(^6\) We have

\[
\sum_{m=1}^{n} \left( F(m) - \frac{1}{x} \log \log n \right)^2
\]

\[
= \sum_{m=1}^{n} F^2(m) - \frac{2}{x} \log \log n \sum_{m=1}^{n} F(m) + n \left( \frac{\log \log n}{x} \right)^2.
\]

Now

\[
\sum_{m=1}^{n} F^2(m) = (1 + o(1))n \left( \frac{\log \log n}{x} \right)^2.
\]

We omit the proof of (15), it is similar to the proof of (14). Thus

\[
\sum_{m=1}^{n} \left( F(m) - \frac{1}{x} \log \log n \right)^2 = o(n(\log \log n)^2)
\]

which proves Theorem 9.

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THEOREM 10. For almost all \( n \) we have
\[
\sum_{p \mid n} b_i = (1 + o(1)) \log \log n \log \log \log n.
\]

THEOREM 11. Let \( 1 < x \) be any number. For almost all \( n \) there exist intervals \((m, m^2)\), \( m^2 \leq n \), such that for every \( m \leq y \leq m^2, n \equiv 0 \pmod{y} \).

We omit the proofs of Theorems 10 and 11. They are similar to that of Theorem 9.

For some time I have not been able to decide the following question: Is it true that almost all integers \( n \) have divisors \( d_1 \) and \( d_2 \), such that \( d_1 < d_2 < 2d_1 \).

(4) Let \( f(n) \) be an additive function which has a distribution function. Then it is well known that
\[
\sum_p \frac{f(p)}{p} < \infty, \quad \sum_p \left(\frac{f(p)}{p}\right)^2 < \infty,
\]
\( f(p) = f(p) \) if \(|f(p)| \leq 1 \) and \( f(p) = 1 \) if \(|f(p)| > 1 \). Assume now that \(|f(p^a)| \leq C \) (\( f(n) \) is assumed to be real valued). We prove the following theorem.

THEOREM 12. Let \(|f(p^a)| \leq c\). Denote by \( F(x) \) the distribution function of \( f(x) \). We have
\[
F(x) > 1 - \exp(-cx),
\]
for every \( c \) and sufficiently large \( x \). In other words the density of integers with \( f(n) \geq x \) is less than \( \exp(-cx) \).

Put \( g(n) = \exp(2cf(n)) \), \( g(n) \) is multiplicative and clearly has a distribution function. Define
\[
f_k(n) = \sum_{\substack{p \mid n, p \leq k}} f(p), \quad g_k(n) = \exp(2cf_k(n)).
\]

For sake of simplicity we assume that \( f(p^a) = f(p) \). It is well known that the distribution function \( F_k(x) \) of \( f_k(n) \) converges to \( F(x) \), thus the distribution function \( G_k(x) \) of \( g_k(x) \) converges to \( G(x) \) (\( G(x) \) is the distribution function of \( g(x) \)). Suppose now that Theorem 12 is false, then there exists a constant \( c \) and infinitely many \( x_r \) with \( x_r \to \infty \) and
\[
F(x_r) > 1 - \exp(-cx_r).
\]
Therefore for any \( r \) there exists a \( k \) so large that
\[
F_k(x_r) > 1 - \exp(-cx_r).
\]

\footnote{P. Erdős and A. Wintner, Amer. J. Math. vol. 61 (1939) pp. 713-721.}
Thus the density of integers with \( g_k(n) > \exp(2cx_n) \) is greater than \( \exp(-cx_n) \) and hence
\[
\sum_{m \leq n} g_k(m) > (1 - \epsilon) \exp(cx_r) \cdot n
\]
for \( n \) sufficiently large. Thus for any \( A \) there exists \( k \) and \( n_0 \), such that for all \( n > n_0 \)
\[
\sum_{m \leq n} g_k(m) > An.
\]
On the other hand
\[
\sum_{m \leq n} g_k(m) = \sum_{m=1}^{n} \prod_{p|m} g_k(p) = \sum_{m=1}^{n} \prod_{p|m} (1 + (g_k(p) - 1)).
\]
Put \( g_k(p) - 1 = h_k(p) \). Clearly
\[
\sum_{m=1}^{n} g_k(m) = \sum_{m=1}^{n} \prod_{p|m} (1 + h_k(p)) = \sum_{d} \left[ \frac{n}{d} \right] h_k(d)
\]
where \( h_k(d) = \prod_{p|d} h_k(p) \). Thus
\[
\sum_{m=1}^{n} g_k(m) \leq n \sum_{d} \frac{h_k(d)}{d} = n \prod_{p} \left( 1 + \frac{h_k(p)}{p} \right).
\]
From the fact that \( g(n) \) has a distribution function and that \( f(p^n) \) is bounded, it easily follows that (we shall give the details in the proof of Theorem 13)
\[
\sum_{p} \frac{h(p)}{p} < \infty, \quad \sum_{p} \frac{(h(p))^2}{p} < \infty, \quad h(p) = g(p) - 1.
\]
Thus finally
\[
\sum_{m=1}^{n} g_k(m) < c_1 n \prod_{p} \left( 1 + \frac{h(p)}{p} \right) < c_2 n,
\]
which contradicts (17), and this contradiction establishes the theorem.

It is easy to see that Theorem 12 is best possible. Let \( \phi(x) \) tend to infinity arbitrarily slowly; then there exists an additive function \( f(n) \) such that its distribution function \( F(x) \) satisfies \( F(x_i) < 1 - \exp(-\phi(x_i)x_i) \) for an infinite sequence \( x_i \) with \( x_i \to \infty \). We omit the proof.

**Theorem 13.** Let \( g(n) \geq 0 \) be multiplicative. Then the necessary and sufficient condition for the existence of a distribution function is that
(18) \[ \sum_{p} \frac{(g(p) - 1)^{1'}}{p} < \infty, \quad \sum_{p} \frac{(g(p) - 1)^{1''}}{p} < \infty, \]

where \( (g(p) - 1)^{1'} = g(p) - 1 \) if \(| g(p) - 1 | \leq 1 \) and 1 otherwise.

The proof follows very easily from (16). Put \( \log(g(n)) = f(n) \). \( g(n) \) has a distribution function if and only if \( f(n) \) has a distribution function. Thus from (16)

(19) \[ \sum_{p} \frac{(\log g(p))'}{p} < \infty, \quad \sum_{p} \frac{(\log g(p))''}{p} < \infty. \]

Now it follows from (19) that if we neglect a sequence of primes \( q \) with \( \sum 1/q < \infty \) that \( |g(p) - 1| < 1/2 \). Thus

\[ \log g(p) = \log (1 + (g(p) - 1)) = g(p) - 1 + (1/2)(g(p) - 1)^{2} + \cdots. \]

Also simple computation shows that \( (\log g(p))^{1'} > (1/4) (g(p) - 1)^{2} \).

Thus from (19)

\[ \sum_{p} \frac{(g(p) - 1)^{2}}{p} < \infty \]

and

\[ \sum_{p} \frac{((1/2) (g(p) - 1)^{2} + (g(p) - 1)^{3} + \cdots)}{p} < \infty. \]

Thus \( \sum_{p}(g(p) - 1)/p < \infty \), which shows that (18) is necessary.

If the two series in (18) converge, then clearly

\[ \sum_{p} \frac{\log g(p)}{p} = \sum_{p} \left( \frac{(g(p) - 1)}{p} + \frac{(1/2)(g(p) - 1)^{2}}{p} + \cdots \right) < \infty \]

and

\[ \sum_{p} \frac{(\log g(p))^{2}}{p} < c \sum_{p} \frac{(g(p) - 1)^{2}}{p} < \infty, \]

which shows that \( f(n) \), and therefore \( g(n) \), has a distribution function. Thus (18) is necessary, which completes the proof of Theorem 13.

These results suggest that if \( g(n) \) is multiplicative, satisfies (18), \( |g(p^n)| < c \), then \( g(n) \) has a mean value, that is, \( \lim (1/x)\sum_{n=1}^{x} f(n) \) exists. I have not yet been able to prove this.