THE SPACE $L^\infty$ AND CONVEX TOPOLOGICAL RINGS

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1. Introduction. The motive for investigating the class $L^\infty$ of functions belonging to all $L^p$-classes has no measure-theoretic origin: it was our desire to discover whether or not in every convex metric ring $R$ one could find a system $\{U\}$ of convex neighborhoods of 0 having the property that $f, g \in U$ implies $fg \in U$. We show here that $L^\infty$ has no proper convex open set $U$ containing 0 and satisfying the relation $UU \subseteq U$, thus supplying the desired counter-example.

The significance of neighborhood systems of the type $\{U\}$ described above is made somewhat clearer by a proof that they insure the existence and continuity of entire functions (for example, the exponential function) on the topological ring $R$.

Such neighborhood systems $\{U\}$ are always present in rings of continuous real-valued functions over any space, provided that convergence means uniform convergence on compact sets.

We also consider the relation of $L^\infty$, $L^\infty$, and the $L^p$-classes, since $L^\infty$ does not seem ever to have been discussed as a topological and algebraic entity.

2. Notation and elementary facts. Let us consider measurable functions defined on $[0, 1]$. For $p \geq 1$ we shall consistently employ the usual notation

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$^1$ More precisely, metrizable, convex, complete topological linear algebra. For these one requires continuity in both ring operations and scalar multiplication. It will appear that $L^\infty$ has these properties.
even when the right side is infinite.

Therefore $L^p$ consists of all functions $f$ for which $\|f\|_p$ is less than $\infty$. $L^\infty$ evidently consists of all functions $f$ for which $\|f\|_1, \|f\|_2, \ldots$, $\|f\|_p, \ldots$ are all finite.

Because of the relation
\[ \|fg\|_p \leq \|f\|_q \|g\|_r, \quad 1/p = 1/q + 1/r, \]
one has
\[ \|f\|_1 \leq \|f\|_2 \leq \ldots, \]
since the measure of $[0, 1]$ is 1. Therefore we may take the sets of functions $f$,
\[ \|f\|_p < \epsilon \]
where $p \geq 1$ and $\epsilon > 0$, as neighborhoods of 0 in $L^\infty$. These neighborhoods are convex because
\[ \|\lambda f + \mu g\|_p \leq \lambda \|f\|_p + \mu \|g\|_p < \epsilon \]
when $\lambda, \mu \geq 0$, $\lambda + \mu = 1$, and $\|f\|_p, \|g\|_p < \epsilon$. Therefore addition is continuous in $L^\infty$ and, by relation (H), multiplication is also.

Multiplication is not generally possible in $L^p$.

Now the inequalities above imply that the limit
\[ \lim_{p \to \infty} \|f\|_p = \|f\|_\infty \]
always exists. (It may be infinite.) Those $f$'s for which $\|f\|_\infty$ is finite form a set usually called $L^\infty$, and $\|f\|_\infty$ is taken as a norm in $L^\infty$. We shall employ the known fact that $\|f\|_\infty$ is also the least number $h$ such that $|f(x)| > h$ only on a set of measure zero.

Multiplication in $L^\infty$ is continuous, since
\[ \|fg\|_\infty \leq \|f\|_\infty \|g\|_\infty, \]
from which it follows that if $U$ is any sphere about 0, contained in the unit sphere of $L^\infty$, then $UU \subseteq U$.

3. The relation of $L^\infty$, $L^\infty$, and $L^p$. These spaces are related by successive proper inclusion.

Theorem 1. $L^\infty \subseteq L^\infty \subseteq L^p$ but $L^\infty \neq L^\infty \neq L^p$. The identity mappings

\[ \text{Cf. E. J. McShane, Integration, Princeton, 1944, for most of the facts which we assume. A formula equivalent to (H) appears on p. 186.} \]
\(L^\infty \to L^\infty \to L^p\) are continuous, but their inverses are not. \(L^\infty\) is dense in \(L^u\), and \(L^u\) is dense in each \(L^p\).

**Proof.** The inclusions and the continuity of the mappings are obvious.

If we define \(l(x) = |\log x|\), then \(l\) does not belong to \(L^\infty\). Since \(||l||_p = (p!)^{1/p}\), \(l \in L^p\) for each \(p \geq 1\), and hence \(l \in L^u\). Thus \(L^u \neq L^\infty\).

Similarly, the function with values \(x^{-1/2p}\) belongs to \(L^p\), but not to \(L^{2p}\), and hence not to \(L^u\).

Now let \(l_n(x) = n^{-1}|\log x|\) or \(n\), whichever is the smaller. Then \(||l_n - 0||_p < n^{-1}||l||_p\), which tends to zero as \(n \to \infty\); but \(||l_n - 0||_\infty = n\), \(n \to \infty\). Thus the inverse of the mapping \(L^\infty \to L^u\) is not continuous.

A similar process applied to the function \(x^{-1/2p}\) yields a sequence which converges to zero in \(L^p\) but not in \(L^{2p}\), and thus not in \(L^u\).

Finally, suppose \(f \in L^u\) be given. Define

\[
f_n(x) = \begin{cases} 
-n & \text{when } f(x) < -n, \\
f(x) & \text{when } -n \leq f(x) \leq n, \\
n & \text{when } n < f(x).
\end{cases}
\]

Then \(f_n \to f\) in each \(L^p\) and hence in \(L^u\). Since the \(f_n\) are taken from \(L^\infty\) the latter is dense in \(L^u\) and in each \(L^p\), which establishes the third sentence of the theorem.

\(L^u\) can be metrized, so as to be complete, by

\[
(f, g) = \sum_{p=1}^{\infty} 2^{-p} \frac{||f - g||_p}{1 + ||f - g||_p}.
\]

4. **Multiplication in \(L^u\).** By relation (H), this is continuous. The following theorem shows the divergence between its properties and those of normed rings.

**Theorem 2.** \(L^u\) is a convex metric commutative ring with the property that if \(U\) is a convex open set in \(L^u\) containing 0, and if \(UU \subset U\), then \(U\) coincides with the whole space \(L^u\).

**Proof.** There exists a \(p \geq 1\) and an \(\varepsilon > 0\) such that \(||f||_p \leq \varepsilon\) implies \(f \in U\). Therefore a function \(f\) having values not greater than \(h\) on a set of measure not greater than \((\varepsilon/h)^p\), and vanishing elsewhere, must lie in \(U\), together with all its powers \(f^2, f^3, \ldots\).

Let \(h = 2\), and set \(m = (\varepsilon/2)^p\), for brevity.

Consider any function \(g\) which has the value \(b\) on a set \(S\) of measure \(a\), and vanishes elsewhere. Suppose \(k\) is any integer such that \(a \leq mk\). Select an integer \(n\) such that \(bk \leq 2^n\). Now we can cover \(S\) by \(k\) nonoverlapping subsets of measure not greater than \(m\) and define
functions $f_1, \cdots, f_k$, where $f_i$ has the value $(bk)^{1/n}$ on the $i$th subset of $S$, and vanishes elsewhere. Thus $f_1, \cdots, f_k \in U$, and also $f_1^n, \cdots, f_k^n \in U$. Since $U$ is convex

$$g = \frac{1}{k} f_1^n + \cdots + \frac{1}{k} f_k^n$$

must belong to $U$.

Now any function $g'$ assuming only a finite number of values is a linear combination, with positive constants whose sum is 1, of such functions as $g$. Therefore these functions lie in $U$.

Since these functions $g'$ are known to be dense in $L^\infty$ and thus in $L^\omega$, we have $U$ a dense, open convex set in $L$. Thus $U = L^\omega$.

**COROLLARY.** The topology assigned to $L^\omega$ cannot be defined by any norm.

5. **Entire functions in rings.** Of course Theorem 2 shows more about $L^\omega$ than is needed for a counter-example to the proposition mentioned in the introduction, as will appear from the following theorem, and the fact that $e^{1/\log x} = 1/x$ is not summable, while $|\log x|$, as we have seen, lies in $L^\omega$.

**THEOREM 3.** If $R$ is a complete topological ring with a complete system $\{U\}$ of convex neighborhoods of zero each satisfying $UU \subset U$, and

$$P(z) = a_0 + a_1 z + a_2 z^2 + \cdots$$

is a power series representing an entire function, then, for each $f \in R$,

$$P(f) = a_0 + a_1 f + a_2 f^2 + \cdots$$

converges, and $P$ is a continuous operation on $R$ into itself.

In particular, for the exponential function, if $U$ is convex, contains zero, and $UU \subset U$, then

$$e^U \subset 1 + 2U.$$

**PROOF.** Let us first show that $P(f)$ converges. Therefore, suppose $U$ is any neighborhood of the system $\{U\}$. Let $f \in R$.

Then for some $t > 0$, $tf \in U$. Hence $(tf)^2, (tf)^3, \cdots$ will all lie in $U$. Further, let us find $m_0$ so large that for $m \geq m_0$

$$|a_m t^{-m}| + |a_{m+1} t^{-m-1}| + \cdots$$

is less than 1. Then, since $U$ is convex, we can deduce that for $n > m > m_0$,

$$a_m t^{-m} (tf)^m + \cdots + a_n t^{-n} (tf)^n$$
or its equivalent

\[ a_m f^m + \cdots + a_n f^n \]

must lie in \( U \).

Since \( R \) is assumed complete, \( P(f) \) converges to a limit.

The continuity of \( P \) can be proved as follows:

\[
D = P(f + h) - P(f) = \sum_{n=0}^{\infty} g_{n+1} h_{n+1}
\]

where

\[
g_n = (f + h)^{n+1} - f^{n+1}.
\]

Let \( U \) be a neighborhood of the system \( \{ U \} \), and suppose \( f/t \in U \) where \( 0 < t < \infty \). Select a real number \( a \),

\[
a > |a_1| (t + 1) + |a_2| (t + 1)^2 + \cdots, \quad a \geq 1,
\]

and require \( h \) to be so close to zero that \( ah \in U \).

There is no point in writing down the expansion of \( g_n \) since terms cannot be collected when \( R \) is not commutative. However, each term will contain \( h \), and if \( g_n \) is written as a sum of products of powers of \( f/t \) and \( h \), the coefficients will add up to \( (t+1)^n - t^n \).

Since \( f/t \) and \( ah \) lie in \( U \), and \( UU \subseteq U \), we have

\[
h_n = (t + 1)^{-n} a g_n \in U,
\]

where, before dividing, we have replaced \( (t+1)^n - t^n \) by \((t+1)^n\). Now \( D \) is a linear combination of \( h_1, h_2, \cdots \) with coefficients whose absolute values add up to less than \( 1 \), and since \( U \) is convex we conclude \( D \subseteq U \).

Therefore \( P \) is continuous at \( f \).

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