ON THE SINGULARITIES OF A CLASS OF FUNCTIONS
ON THE UNIT CIRCLE

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Pólya has suggested and Szegö and others have proved the following theorem.¹

THEOREM. Let \( f(z) \) be a function regular in the whole plane including \( z = \infty \) except at \( z = 1 \). Let

\[
\begin{align*}
  f(z) &= \begin{cases} 
    \sum a_n z^n, & |z| < 1, \\
    \sum b_n / z^n, & |z| > 1. 
  \end{cases}
\end{align*}
\]

If \( a_n = O(n^k) \) and \( b_n = O(n^k) \) then \( f(z) \) is a rational function.

The above theorem is generalized in this paper as follows.

THEOREM 1. Let \( f(z) \) be regular in the whole plane including \( z = \infty \), except possibly at a certain set \( S \) of points on \( |z| = 1 \) (the set \( S \) being not everywhere dense on the complete circumference of the unit circle). Let

\[
\begin{align*}
  a_n &= O(n^k), \quad b_n = O(n^k), \\
  m &\sim \left| \frac{1}{n} \right|, \quad |l| > 1,
\end{align*}
\]

and let \( a_n = O(n^k) \), \( b_n = O(n^k) \); then the following results hold.

(i) Every isolated singularity on \( |z| = 1 \) will be a pole of order not exceeding \( k + 1 \).

(ii) If there are only a finite number of singularities on \( |z| = 1 \), then \( f(z) \) is a rational function.

THEOREM 2. There exists a function satisfying the hypothesis of Theorem 1 and having an infinite number of singularities on the unit circle; also there exists a function satisfying the same hypothesis and having no isolated singularities.

LEMMA 1. Let \( f(z) \) be an integral function and let

\[
I_p(r) = \int_0^{2\pi} |f(re^{i\phi})|^p d\phi
\]

be bounded on a sequence of circles \( r = r_n \) tending to infinity, for some \( p > 0 \). Then \( f(z) \) reduces to a constant.

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PROOF. \(|f(z)|^p\) is subharmonic in any region of the \(z\)-plane. By Poisson’s integral formula

\[
|f(z)|^p \leq \frac{1}{2\pi} \int_0^{2\pi} \frac{(R^2 - r^2)^p}{R^2 + r^2 - 2rR \cos (\phi - \theta)} \, p d\phi
\]

where \(|z| = r < R = r_n\). Hence

\[
|f(z)|^p \leq \frac{1}{2\pi} \frac{R^2 - r^2}{(R - r)^2} \int_0^{2\pi} |f(Re^{i\phi})| p d\phi \leq K \frac{R + r}{R - r}.
\]

Putting \(R = 2r\) we get

\[
|f(z)|^p \leq 3K/2 \quad \text{on} \quad |z| = R/2.
\]

Hence \(f(z)\) is bounded on \(|z| = r_n/2\) and so it reduces to a constant.

**Lemma 2.** Let \(f(z)\) be regular for \(|z| \geq R\) except probably at infinity; and let

\[
\int_0^{2\pi} |f(z)|^p d\phi
\]

be bounded on a sequence of circles \(|z| = r_n\) tending to infinity. Then \(f(z)\) is regular at infinity.

**Proof.** We can write \(f(z) = g(z) + h(z)\) where \(g(z)\) is an integral function and \(h(z)\) regular at infinity. Since \(h(z)\) is bounded at infinity, it follows from Minkowski’s inequality when \(p > 1\) (and still simpler when \(p \leq 1\)) that \(\int_0^{2\pi} |g(z)|^p d\phi (z = re^{i\phi})\) is bounded on a sequence of circles \(r = r_n\) tending to infinity. Hence \(g(z)\) is constant by Lemma 1 and so \(f(z)\) is regular at infinity.

**Proof of Theorem 1.** To prove that every isolated singularity will be a pole, it is enough to prove that if \(z = 1\) is an isolated singularity, it is a pole since every other singularity can be brought to \(z = 1\) by a rotation. We suppose that \(k\) is a positive integer. We have already supposed that \(z = 1\) is an isolated singularity of \(f(z)\). Let \(x = (1+z)/(1-z)\). This transforms the unit circle in the \(z\)-plane into the imaginary axis in the \(x\)-plane and \(z = 1\) corresponds to \(x = \infty\). The function \(\phi(x) = f(z)\) given by the above relation is therefore regular for \(|x| \geq R_0\) (where \(R_0\) is some number), except at \(x = \infty\).

From our assumptions about the coefficients we obtain

\[
|f(z)| \leq \frac{c}{|1 - z|^{k+1}}
\]

in the neighbourhood of the circle \(|z| = 1\). Hence
Let \( \psi(x) = (1 - z)^{k+1}f(z) \) where \( z = (x - 1)/(x + 1) \). Then

\[
|\psi(x)| \leq \frac{2^{k+1}}{x + 1} \left| \frac{1 - (x - 1)/(x + 1)}{x - 1} \right|^{k+1}
\]

\[
\leq \frac{c_1}{\left| x + 1 - |x - 1| \right|^{k+1}}.
\]

Let \( x = \rho e^{i\gamma} \). Then \(|x + 1|^2 - |x - 1|^2 = 4\rho \cos \gamma \). Hence

\[
|\psi(x)| \leq \frac{c_1 \left\{ |x + 1| + |x - 1| \right\}^{k+1}}{\left\{ |x + 1|^2 - |x - 1|^2 \right\}^{k+1}}
\]

\[
\leq c_1 \left\{ \frac{|x + 1| + |x - 1|}{4\rho \cos \gamma} \right\}^{k+1}
\]

\[
\leq \frac{c_2}{\cos \gamma} \left| x + 1 - |x - 1| \right|^{k+1}
\]

if \( \rho \geq \rho_0 \) is sufficiently large. Hence

\[
|\psi(x)|^{1/2(k+1)} \leq \frac{c_3}{\cos \gamma}^{1/2}
\]

except when \( \gamma = \pm \pi/2 \). Hence

\[
\int_0^{2\pi} |\psi(x)|^{1/2(k+1)} d\gamma \leq \int_0^{2\pi} \frac{c_3 d\gamma}{(\cos \gamma)^{1/2}}
\]

which is a convergent integral. Hence

\[
\int_0^{2\pi} |\psi(x)|^{1/2(k+1)} d\gamma
\]

is bounded and so, by Lemma 2, \( \psi(x) \) is regular at infinity. Hence \((1 - z)^{k+1}f(z)\) is regular at \( z = 1 \), which shows that \( f(z) \) has a pole of order not exceeding \( k+1 \) at \( z = 1 \).

This proves (i). To prove (ii) it follows by part (i) that each of the finite number of singularities on \( |z| = 1 \) is a pole. Hence \( f(z) \) is a regular function throughout the \( z \)-plane including infinity, except for a finite number of poles. Hence \( f(z) \) is a rational function.
PROOF OF THEOREM 2. Consider the function

\[ f(z) = \sum_{1}^{\infty} \frac{1}{2^n} \frac{1}{(z - \alpha_n)} \]

where \((\alpha_n)\) is any sequence of points on \(|z| = 1\). If the sequence \((\alpha_n)\) has only one limit point the above function \(f(z)\) has a pole at each of these points \(\alpha_n\) and an essential singularity at the limit point of the sequence \((\alpha_n)\). It is regular elsewhere. If

\[ f(z) = \sum_{0}^{\infty} a_p z^p \quad \text{for } |z| < 1 \]

then

\[ a_p = -\sum_{n=1}^{\infty} \frac{1}{2^n} \frac{1}{\alpha_n^{p+1}} \]

and therefore \(|a_p| \leq 1\). Similarly if \(f(z) = \sum b_p z^p \ (|z| > 1)\) then \(|t_p| \leq 1\). Hence \(a_p\) and \(b_p\) are certainly \(O(n^k)\) for any \(k \geq 0\). To prove the second part, it is enough to take \((\alpha_n)\) in the above example to be everywhere dense on some arc of the unit circle, the arc not being the whole of the circumference. The function \(f(z)\) will have a non-isolated essential singularity at every point of this arc and the coefficients \(a_n\) and \(b_n\) are bounded.

This example shows that the part (ii) of Theorem 1 is in a sense the best possible result, for the function constructed satisfies the conditions on the coefficients while it is not a rational function since it has an infinite number of singularities on the unit circle.

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