ON A GENERALIZATION OF THE STIELTJES MOMENT PROBLEM

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The "generalised moment problem"

\( \int_0^\infty t^n d\alpha(t) = \mu_n \quad (0 = \lambda_0 < \lambda_1 < \lambda_2 \cdots < \lambda_n \to \infty) \)

is said to be determined if there is at most one increasing function \( \alpha(t) \) satisfying (1) and normalized by \( \alpha(0) = 0 \). R. P. Boas, Jr., who first considered this problem \([1]\) gave conditions under which (1) is determined. These do not include the best known result in the classical case \( \lambda_n = n \), namely Carleman's criterion: If \( \lambda_n = n \) and \( \sum \mu_n^{-1/2n} = \infty \), then (1) is determined. I shall now prove a theorem including Carleman's test as a special case. On the other hand this theorem will not include the results of Boas, as I shall from now on assume

\( \lambda_{n+1} - \lambda_n > c \quad (n = 1, 2, \cdots) \)

for some \( c > 0 \).

Let

\[ \psi(r) = \exp \left\{ \sum_{0 < \lambda_r \leq r} \lambda_r^{-1} \right\}. \]

THEOREM. If there are a non-increasing function \( \phi(r) \) and positive constants \( A \) and \( a \) such that

\[ \psi(r) > A (r/\phi(r))^a \]

and if

\[ \sum_{2}^{\infty} \frac{\lambda_n - \lambda_{n-1}}{\mu_1^{1/\alpha_n} \phi(\lambda_{n-1})} = \infty, \]

then (1) is determined.

The proof is based on the following lemma.

LEMMA. If (2) is the case, then

\[ G(z) = \prod_{r=1}^{\infty} \frac{\lambda_r + z}{\lambda_r - z} \frac{e^{-2z/\lambda}} {e^{-2z/\lambda}} \]

Received by the editors May 29, 1946.

\(^1\) Numbers in brackets refer to the references cited at the end of the paper.
is regular apart from poles at the $\lambda$, and for some constant $B$

$$|G(z)| < B^s(\psi(r))^{-s} \quad (z = x + iy = re^{i\theta})$$

in $x \geq 0$ except in circles of radius $c/3$ round the $\lambda_v$.

This lemma is proved in [2].

**Proof of the Theorem.** We must prove that two increasing solutions, $\alpha_1(t)$ and $\alpha_2(t)$, of (1) can differ by a constant only. Consider

$$F(z) = \frac{1}{2} \int_0^\infty t^s d(\alpha_1 - \alpha_2).$$

$F(z)$ is regular in $\mathfrak{R}z = x > 0$ and

$$|F(z)| < \frac{1}{2} \int_0^\infty t^s d(\alpha_1 + \alpha_2) \leq (v(x))^{s^2},$$

say. Since $(\int_0^\infty t^s d(\alpha_1 + \alpha_2)/\int_0^\infty d(\alpha_1 + \alpha_2))^{1/s}$ is an increasing function of $x$, by Hölder’s inequality, we may choose

$$v(x) = K\mu_n^{1/\lambda_n} \quad (\lambda_{n-1} < x \leq \lambda_n).$$

Also $F(\lambda_n) = 0$ ($n = 1, 2, \cdots$), but unless $\alpha_1(t) - \alpha_2(t) =$ const., $F(z)$ does not vanish identically. It is therefore sufficient to prove that $F(z)$ is identically zero.

If $G(z)$ is the function defined in the lemma, let

$$H(z)s^{-s} = F(z/a)G(z/a)z^se^{-G(1+s)}(1 + s)^{-s}z^{-s}.$$

This function is regular in $\mathfrak{R}z > 0$. Also, if $z = x + iy = re^{i\theta}$

$$|F(z/a)G(z/a)| \leq (v(x/a)\phi(r/a)BA^{-1}ar^{-1})^s \leq (v(x/a)\phi(x/a)BA^{-1}ar^{-1})^s,$$

$$|z^s| = r^se^{-r\theta \sin \theta} \leq r^se^{-r|\theta|/2}z^s,$$

since $\theta \sin \theta = \pm \sin \theta |\sin \theta|/2 - \cos \theta$ for $|\theta| \leq \pi/2$;

$$s^{-s} = |s|^{-s}e^{sv\arg s}.$$

Therefore

(5) \quad $$|H(z)s^{-s}| < (v(x/a)\phi(x/a) |s|^{-1})^s e^{-(\pi/2 - |\arg z|/2)(1 + r)^{-2}},$$

provided that $C$ is taken sufficiently large. Consider now

(6) \quad $$g(s) = \int_{1-i\infty}^{1+i\infty} H(z)s^{-s}dz.$$
Because of (5) the integral is uniformly convergent for $|s| \geq 1$, $|\arg s| \leq \pi/2$. In particular $g(s)$ is a regular function of $s$ in $|s| > 1$, $|\arg s| < \pi/2$. It also follows from (5) that the line of integration in (6) may be shifted to any other line $x = b > 0$. Taking $b = \xi$ and using (5) gives

$$|g(s)| < 2(\nu(a)\phi(\xi/a))^{\xi} |s|^{-\xi}$$

for every $\xi > 0$.

By a theorem due to Carleman and Ostrowski (7) implies that $g(s)$ vanishes identically, if

$$\int_1^\infty (\nu(a)\phi(\xi/a))^{-1}d\xi = \infty$$

(see [3], in particular Satz IV and §14).

By (4)

$$\int_{a_{n-1}}^{a_n} (\nu(a)\phi(\xi/a))^{-1}d\xi \geq a \frac{\lambda_n - \lambda_{n-1}}{\mu_{n}^{1/a_n}\phi(\lambda_{n-1})}$$

so that (3) implies (8). Therefore $g(s)$ vanishes identically. By a well known uniqueness theorem for the Mellin transform this implies that $H(z)$ is zero and so $F(z)$ must be equal to zero everywhere. Q.e.d.

References


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