1. Introduction. Consider a Taylor series \( f(z) = \sum_{n=0}^{\infty} a_n z^n \) with radius of convergence unity. Let the coefficients \( a_n \) be the values taken on by a regular function \( a(z) \) for \( z = 0, 1, \ldots \).

The object of this paper is to study the Taylor series under the assumption that \( a(z) \) is regular in certain domains.\(^1\) The results obtained are of the nature of domains in which the function defined by \( \sum_{n=0}^{\infty} a_n z^n \) is regular and of domains which contain the singularities of the function defined by the series. In terms of \( a(z) \) fairly general sufficient conditions are given such that the circle of convergence is not a cut for the function.

The results may be regarded as a generalization of a theorem due to LeRoy and Lindelöf.\(^2\)

**Theorem (LeRoy and Lindelöf).** Suppose (a) \( a(x + iy) \) is regular in the semiplane \( x \geq \alpha \), (b) there is a \( \theta < \pi \) such that for every arbitrary small positive \( \epsilon \) and for sufficiently large \( \rho \)

\[
| a(\alpha + \rho \exp(i\phi)) | < \exp[\rho(\theta + \epsilon)], \quad -\pi/2 \leq \psi \leq \pi/2.
\]

Then

\[
f(z) = \sum_{n=0}^{\infty} a(n)z^n, \quad z = r \exp(i\phi)
\]

is regular in the angle

\[
\theta < \phi < 2\pi - \theta.
\]

The generalization of this theorem that we prove consists, under suitable restrictions, in replacing the semiplane \( x \geq \alpha \) by an angular opening including the axis of positive reals in its interior.

The singularities of the function \( f(z) \) studied in this paper are those of a “principal branch” obtained by immediate continuation of the series.

Consider an angular opening with vertex on the positive real axis which includes the axis of reals in its interior. Suppose \( a(z) \) has no singularities in this angular opening with the possible exception of
the point at infinity. Let the sides of the angular opening make angles \( \psi_1 \) and \( \psi_2 \) with the axis of reals.

Our problem is to characterize the behavior of the function \( f(z) \) in terms of the magnitudes of the angles \( \psi_1 \) and \( \psi_2 \) and the type of singularity \( a(z) \) has at infinity.

![FIG. 1](image)

We consider first the case where \( a(z) \) may have a pole of order \( K \) at infinity and then the case in which infinity may be an essential singularity for \( a(z) \).

2. \( a(z) \) may have a pole at infinity. Suppose now that \( a(z) \) is regular interior to and on the sides of the angular opening shown in Figure 1, except for at most a pole of order \( K \) at infinity. In Figure 1 let \( l - 1 < h < l \) where \( l \) is a positive integer.

By the calculus of residues if \( F(\omega) \) and \( G(\omega) \) are uniform functions
in a domain and if $G(\omega)$ has only simple zeros $\alpha_i$ in this domain, then (integration being understood in the positive sense)

\[ \int_C \frac{F(\omega)}{G(\omega)} \, d\omega = \frac{F(\alpha_i)}{G'(\alpha_i)} \]

where $C$ is a path enclosing $\alpha_i$ but no other zero of $G(\omega)$ and $G'(\alpha_i)$ is the derivative of $G(\omega)$ evaluated at $\omega = \alpha_i$.

Let $F(\omega) = a(\omega)z^\omega$ and $G(\omega) = \exp(2\pi i \omega) - 1$.

For a given value of $z = r \exp(i\theta)$ we shall understand by $z^\omega$

\[ z^\omega = \exp[\omega(\log r + i\theta)], \quad 0 \leq \theta < 2\pi. \]

Here $\exp z$ is the principal value of $e^z$. This convention will be adhered to throughout the paper.

Then by (1)

\[ \frac{1}{2\pi i} \int_C \frac{a(\omega)z^\omega}{\exp(2\pi i \omega) - 1} \, d\omega = a(n)z^n \]

where $C$ is a path enclosing $n$ and no other real integer. $a(\omega)$ is a function regular in the angular opening. This choice of $F(\omega)$ and $G(\omega)$ led to the well known theorem of LeRoy and Lindelöf [1] and is of course a well known method for the summation of certain series. The analysis of this paper follows lines similar to those in the analysis of Lindelöf.

Consider

\[ \int_{C_{h,R}} \frac{a(\omega)z^\omega}{\exp(2\pi i \omega) - 1} \, d\omega \]

where $C_{h,R}$ is the path formed by the two sides of the angular opening, with vertex $h > 0$, and the arc of a circle of radius $R$, where $R$ is a positive integer. This is indicated in Figure 1.

By application of (2) it follows that

\[ \frac{1}{2\pi i} \int_{C_{h,R}} \frac{a(\omega)z^\omega}{\exp(2\pi i \omega) - 1} \, d\omega = \sum_{n=-l}^{R+l-1} a(n)z^n. \]

Denote the sides of the angular opening corresponding to $\psi_1$ and $\psi_2$ by $I_1$ and $I_2$ and the arc of a circle by $C$. Then

\[ \int_{C_{h,R}} \frac{a(\omega)z^\omega}{\exp(2\pi i \omega) - 1} \, d\omega = \int_{I_2} \frac{a(\omega)z^\omega}{\exp(2\pi i \omega) - 1} \, d\omega \]

\[ + \int_C \frac{a(\omega)z^\omega}{\exp(2\pi i \omega) - 1} \, d\omega + \int_{I_1} \frac{a(\omega)z^\omega}{\exp(2\pi i \omega) - 1} \, d\omega. \]
It will now be shown that, if we place certain restrictions on \( \psi_1, \psi_2 \) and \( z \),

\[
\int_C \frac{a(\omega)z^\omega}{\exp(2\pi i\omega) - 1} \, d\omega
\]

converges uniformly to zero as \( R \) becomes infinite.

Let \( z = r \exp(i\pi) \) and \( \omega = h + R \exp(i\psi) \) with \(-\pi < \psi \leq \pi < \pi\).

By hypothesis there exists an \( R_0 \) such that in the angular opening with vertex \( h > 0 \)

\[
|a(h + R \exp(i\psi))| < AR^K, \quad R > R_0,
\]

where \( A \) is a constant and \( K \) is a positive integer. On the arc \( C \)

\[
\frac{1}{\exp(2\pi i\omega) - 1} < B,
\]

where \( B \) is a constant. This follows from the fact that \( \omega \) is bounded away from an integer by our choice of \( h, l \) and \( R \).

Clearly

\[
\frac{1}{\exp(2\pi i\omega) - 1} = \frac{1}{1 - \exp(-2\pi i\omega)} \mid \exp(-2\pi i\omega)\mid.
\]

Hence on \( C \),

\[
\frac{1}{\exp(2\pi i\omega) - 1} < D \exp(2\pi R \sin \psi),
\]

where \( D \) is a constant. This inequality proves useful for \(-\pi < \psi \leq 0\).

Now

\[
\left| \int_C \frac{a(\omega)z^\omega}{\exp(2\pi i\omega) - 1} \, d\omega \right| \leq \int_C \left| \frac{a(\omega)}{\exp(2\pi i\omega) - 1} \right| |z^\omega| \, d\omega,
\]

and

\[
|z^\omega| = \mid \exp [(h + R \cos \psi + iR \sin \psi)(\log r + i\pi)] \mid
\]

\[
= r^h \exp [-R(\log r^{-1} \cos \psi + \pi \sin \psi)].
\]

If \( r < 1 \), then

\[
b(r, \psi) = \log r^{-1} \cos \psi + \pi \sin \psi > 0, \quad 0 \leq \psi \leq \pi/2.
\]

If \( r > \exp(\pi \tan \psi) \), then

\[
b_1(r, \psi) = \log r^{-1} \cos \psi + \pi \sin \psi > 0, \quad \pi/2 \leq \psi \leq u < \pi.
\]

Clearly \( r > \exp(\pi \tan u) \) is sufficient for (10) to hold. We note here
and also in (11) below, that the case $|\psi| = \pi/2$ is included. If $r < 1$ then

(11) \[ b_2(r, \psi) = \log r^{-1} \cos \psi - \pi \sin \psi > 0, \quad -\pi/2 \leq \psi \leq 0. \]

If $r > \exp (-\pi \tan \psi)$, then

(12) \[ b_3(r, \psi) = \log r^{-1} \cos \psi - \pi \sin \psi > 0, \quad -\pi < \psi \leq \pi - \pi/2. \]

Clearly $r > \exp (-\pi \tan \psi)$ is sufficient for (12) to hold.

We have seen that if $-\pi/2 \leq \psi \leq \pi/2$, $r < 1$ implies $b(r, \psi) > 0$ and $b_2(r, \psi) > 0$. Denote, for $\alpha > 0$ but otherwise arbitrarily small, by $r_1$ the larger of $\exp(\pi \tan \psi) + \alpha$ and $\exp(-\pi \tan \psi) + \alpha$. Set $r_2 = 1 - \alpha'$, $\alpha' > 0$ but otherwise arbitrarily small. Then $r_1 \leq r \leq r_2$ implies $\pi/2 < u < \pi$ and $-\pi < \psi < -\pi/2$. We choose $\alpha$ and $\alpha'$ sufficiently small, so that $r_1 \leq r \leq r_2$ is an interval consisting of more than one point. Then, for a given $\alpha$ and $\alpha'$ and $r_1 \leq r \leq r_2$, (9), (10), (11), and (12) take on their minimum values on their respective intervals of definition. Let these be $b' > 0$, $b_1' > 0$, $b_2' > 0$ and $b_3' > 0$. Denote by $\beta > 0$ the smallest of these four values. Then by application of (5), (6) and (7) for $R > R_0$, (8) becomes

\[
\left| \int_C \frac{a(\omega)z^\omega}{\exp(2\pi i\omega) - 1} \, d\omega \right| \leq T_0(A, B, D)R \int_u^v \exp(-\beta R) \, d\psi
\]

\[ = T_0(A, B, D)R \exp(-\beta R)(u - v). \]

Given an arbitrary $\varepsilon > 0$, there exists an $R_1$ such that, for all $R > R_0$, $R_1$ and $r_1 \leq r \leq r_2$, the right-hand member of (13) is less than $\varepsilon$. That is

(14) \[ \left| \int_C \frac{a(\omega)z^\omega}{\exp(2\pi i\omega) - 1} \, d\omega \right| < \varepsilon, \quad R > R_0, R_1; r_1 \leq r \leq r_2. \]

By (14) it follows that the integral along the arc converges uniformly to zero as $R$ becomes infinite. Hence by (3) and (4) we may write

(15) \[ \sum_{n=-\infty}^{\infty} a(n)z^n = \int_{I_1} \frac{a(\omega)z^\omega}{\exp(2\pi i\omega) - 1} \, d\omega + \int_{I_1} \frac{a(\omega)z^\omega}{\exp(2\pi i\omega) - 1} \, d\omega. \]

Let

\[ J_{\psi_1}(z) = \int_{I_1} \frac{a(\omega)z^\omega}{\exp(2\pi i\omega) - 1} \, d\omega \]

and

\[ J_{\psi_2}(z) = \int_{I_1} \frac{a(\omega)z^\omega}{\exp(2\pi i\omega) - 1} \, d\omega. \]

It will be shown that $J_{\psi_1}(z) + J_{\psi_2}(z)$ is regular for $z$ in a domain
which contains the interval $-r_2 \leq z \leq -r_1$ in its interior.

In this discussion four cases with regard to the angles $\psi_1$ and $\psi_2$ present themselves naturally.

A: $-\pi/2 < \psi_2 < 0$, $0 < \psi_1 < \pi/2$

B: $-\pi/2 < \psi_2 < 0$, $\psi_1 = \pi/2$

C: $\psi_2 = -\pi/2$, $\psi_1 = \pi/2$

D: $-\pi < \psi \leq \psi_2 \leq -\pi/2$, $\pi/2 < \psi_1 \leq \psi < \pi$.

Consider case A. Here

$$J_{\psi}(z) = \int_{I_1} \frac{a(\omega) z^\omega}{\exp(2\pi i \omega) - 1} d\omega$$

$$= -\int_0^\infty \frac{a(\omega) \exp [(i + R \exp(2i\psi_1))(-\log r + i\theta)]}{\exp(2\pi i \omega) - 1} \exp(i\psi_1)dR.$$  

From (5) and (6) it follows that

$$|J_{\psi_1}(z)| < ABr^h \int_0^\infty R^k \exp \left[ - R(\log r^{-1} \cos \psi_1 + \theta \sin \psi_1) \right] dR.$$  

If

$$0 < r < \exp(\theta \tan \psi_1), \quad 0 \leq \theta < 2\pi,$$

then $b(r, \theta, \psi) = \log r^{-1} \cos \psi_1 + \theta \sin \psi_1 > 0$.

Suppose $z = r \exp(\theta \theta)$ is in some closed domain$^8$ contained in the domain defined by (17). Denote by $b'(\psi_1) > 0$ the minimum assumed by $b(r, \theta, \psi_1)$ in this closed domain. Then (16) becomes

$$|J_{\psi_1}(z)| < ABr^h \int_0^\infty R^k \exp \left[ - b'(\psi_1)R \right] dR < M,$$

where $M$ is a constant.

Therefore $J_{\psi_1}(z)$ converges uniformly for $z$ in any closed domain contained in (17). For $\omega$ on $I_1$ and $z$ contained in the domain defined by (17) the integrand of $J_{\psi_1}(z)$ is continuous in $\omega$ and $z$. It follows from our definition of $z^\omega$ that for a fixed $\omega$ on $I_1$ the integrand is a regular function of $z$ for $z$ in any closed domain contained in (17). Then by well known theorems [2], $J_{\psi_1}(z)$ is regular for $z$ in any closed domain contained in (17).

Consider

$^8$ $D$ is said to be a closed domain if there exists a domain $E$ with the property that $D = E$. Here if $E$ is a given set and $E'$ its derived set then $E = E + E'$.
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\[ J_{\psi_2}(z) = \int_{I_5} \frac{a(\omega)z^\omega}{\exp(2\pi i\omega) - 1} d\omega \]
\[ = \int_0^\infty \frac{a(\omega) \exp \left[ (h + R \exp(i\psi_2))(\log r + i\theta) \right]}{\exp(2\pi i\omega) - 1} \exp(i\psi_2) dR. \]

By (5) and (7) we may write
\[ |J_{\psi_2}(z)| \]
\[ < AD^r \int_0^\infty R^k \exp \left[ - R \log r^{-1} \cos \psi_2 + (\theta - 2\pi) \sin \psi_2 \right] dR. \]

If
\[ 0 < r < \exp \left[ (\theta - 2\pi) \tan \psi_2 \right] \]
then \( b(r, \theta, \psi_2) = \log r^{-1} \cos \psi_2 + (\theta - 2\pi) \sin \psi_2 > 0. \)

Let \( z = r \exp(i\theta) \) be in any closed domain contained in the domain defined by (19). Denote by \( b'(\psi_2) > 0 \) the minimum taken on by \( b(r, \theta, \psi_2) \) in this closed domain; then (18) becomes
\[ |J_{\psi_2}(z)| < AD^r \int_0^\infty R^k \exp \left[ - b'(\psi_2) R \right] dR < N, \]
where \( N \) is a constant.

It then follows by the same analysis employed in the case of \( J_{\psi_1}(z) \) that \( J_{\psi_2}(z) \) is regular in any closed domain contained in the domain defined by (19). The function \( J_{\psi_1}(z) + J_{\psi_2}(z) \) will therefore be regular for \( z = r \exp(i\theta) \) in any closed domain contained in the domain common to (17) and (19).

It has been shown (15) that
\[ \sum_{n=1}^\infty a(n)z^n = J_{\psi_1}(z) + J_{\psi_2}(z) \]
where \( z = r \exp(i\pi) \) and \( r_1 \leq r \leq r_2 \). But \( J_{\psi_1}(z) + J_{\psi_2}(z) \) has been shown to be regular in a domain which includes the interval \( -r_2 \leq z \leq -r_1 \) in its interior. Hence \( J_{\psi_1}(z) + J_{\psi_2}(z) \) provides the analytic continuation of the function defined by \( \sum_{n=1}^\infty a(n)z^n \) to any closed domain contained in the domain common to (17) and (19). Since
\[ f(z) = \sum_{n=0}^{l-1} a(n)z^n + \sum_{n=l}^{\infty} a(n)z^n, \]
it is evident that \( J_{\psi_1}(z) + J_{\psi_2}(z) + \sum_{n=0}^{l-1} a(n)z^n \) provides the analytic continuation of \( f(z) \) to the same domain. Hence in this case \( f(z) \) will
in general be regular in a domain bounded by two spirals as indicated in Figure 2.

Consider case B. By an analysis similar to that given in case A it is easily shown that $J_{\phi_1}(z) + J_{\phi_2}(z)$ is regular in any closed domain contained in the domain common to

\[(20) \quad 0 < r < \exp [(\theta - 2\pi) \tan \psi], \quad 0 \leq \theta < 2\pi,\]

and the whole complex plane excluding the segment 1 to $+\infty$. By an argument similar to that employed in case A it is easily seen that $f(z)$ is regular in any closed domain contained in the domain common to (20) and the whole complex plane excluding the segment 1 to $+\infty$.

Case C. Here we have a very special case of the theorem of LeRoy and Lindelöf [1]. It could be shown by an analysis similar to the preceding that $f(z)$ is regular in any domain of the complex plane excluding the segment 1 to $+\infty$.

Finally we consider case D. By a method similar to that employed in case A it is simple to show that $J_{\phi_1}(z)$ converges uniformly for $z$ in any closed bounded domain contained in the domain defined by

![Fig. 2](image-url)
(21) \[ r > \exp (\theta \tan \psi_1), \quad 0 \leq \theta < 2\pi, \]
and that \( J_{\psi_1}(z) \) converges uniformly for \( z \) in any closed bounded domain contained in the domain defined by

(22) \[ r > \exp [(\theta - 2\pi) \tan \psi_2], \quad 0 \leq \theta < 2\pi. \]

Now the domain common to (21) and (22) contains the segment \(-r_2 \leq z \leq -r_1\) in its interior. This is easily seen by setting \( \theta = \pi \) in (21) and (22) and noting that \( r_1 \) is the larger of \( \exp(\pi \tan u) + \alpha \) and \( \exp(\pi \tan v) + \alpha, \alpha > 0 \). Hence by arguments similar to those used in case A

\[ J_{\psi_1}(z) + J_{\psi_2}(z) + \sum_{n=0}^{l-1} a(n)z^n \]
provides the analytic continuation of \( f(z) \) to any closed bounded domain contained in the domain common to (21) and (22). For the case \( \psi_1 = \psi_2 \) the spirals (21) and (22) are indicated in Figure 3. In calculating \( J_{\psi_1}(z) \) and \( J_{\psi_2}(z) \) for a given \( z = r \exp (i\theta) \) we recall that by our convention

\[
z^\alpha = \exp \left[ \omega (\log r + i\theta) \right], \quad 0 \leq \theta < 2\pi.
\]

Hence we have the following theorem.

**Theorem 1.** Let \( f(z) = \sum_{n=0}^{\infty} a_n z^n \) with radius of convergence unity. Let the coefficients \( a_n \) be the values taken on by an analytic function \( a(z) \) at \( z = 0, 1, 2, \cdots \). Suppose \( a(z) \) is regular with the possible exception of a pole of order \( K \) at infinity, in an angle with vertex \( \theta > 0 \) (non-integral) on the axis of reals and including the axis of positive reals in its interior. Let the sides of this angle make angles \( \psi_1 \) and \( \psi_2 \) with the axis of reals. Then, if \( \gamma > 0 \) but otherwise arbitrarily small, and

- **A:** \( 0 < \psi_1 < \pi/2, \quad -\pi/2 < \psi_2 < 0 \),

\( f(z) \) is regular in the domain common to

\[
r \leq \exp \left[ \theta \tan \psi_1 \right] - \gamma, \quad 0 \leq \theta < 2\pi,
\]

and

\[
r \leq \exp \left[ (\theta - 2\pi) \tan \psi_2 \right] - \gamma, \quad 0 \leq \theta < 2\pi.
\]

- **B:** \( \psi_1 = \pi/2, \quad -\pi/2 < \psi_2 < 0 \),

\( f(z) \) is regular in any closed domain common to

\[
r \leq \exp \left[ (\theta - 2\pi) \tan \psi_2 \right] - \gamma, \quad 0 \leq \theta < 2\pi,
\]

and the whole complex plane excluding the segment 1 to \( +\infty \). If

- **C:** \( \psi_1 = \pi/2, \quad \psi_2 = -\pi/2 \),

\( f(z) \) is regular in any closed domain of the complex plane excluding the segment 1 to \( +\infty \). If

- **D:** \( \pi/2 < \psi_1 \leq u < \pi, \quad -\pi < v \leq \psi_2 < -\pi/2 \),

\( f(z) \) is regular in any bounded domain common to

\[
r \geq \exp (\theta \tan \psi_1) + \gamma, \quad 0 \leq \theta < 2\pi,
\]

and
It is clear\textsuperscript{4} that the theorem above is still true even though \(a(z)\) does not have a pole at infinity. All that is necessary is that \(a(z)\) be single-valued in the angle and that \(\left| a(h + R \exp (i\psi)) \right| < A R^k, R > R_0.\)

3. \(a(z)\) may have an essential singularity at infinity. Suppose \(a(z)\) is regular interior to and on the sides of the angular opening of Figure 1. Suppose there exists an \(R_0\) such that for \(R > R_0\) and \(z = h + R \exp (i\psi)\) in this angle

\[
\left| a(h + R \exp (i\psi)) \right| < \exp (\delta R).
\]

In order to simplify the work to follow suppose \(\delta \leq \pi - d, \ d > 0.\)

It will be shown that, if we place certain restrictions on \(\psi_1, \psi_2\) and \(z,\)

\[
\int \frac{a(\omega)\omega}{\exp (2\pi i\omega) - 1} \, d\omega
\]

converges uniformly to zero as \(R\) becomes infinite. Let \(z = r \exp (i\pi)\) and \(\omega = h + R \exp (i\psi)\) with \(-\pi / 2 \leq \psi \leq \pi / 2.\)

If \(r < \exp \left[ \pi \tan \psi - \delta \sec \psi \right]\), then

\[
\int \frac{a(\omega)\omega}{\exp (2\pi i\omega) - 1} \, d\omega
\]

converges uniformly to zero as \(R\) becomes infinite. Let \(z = r \exp (i\pi)\) and \(\omega = h + R \exp (i\psi)\) with \(-\pi / 2 \leq \psi \leq \pi / 2.\)

If \(r < \exp \left[ -\pi \tan \psi - \delta \sec \psi \right]\), then

\[
\int \frac{a(\omega)\omega}{\exp (2\pi i\omega) - 1} \, d\omega
\]

converges uniformly to zero as \(R\) becomes infinite. Let \(z = r \exp (i\pi)\) and \(\omega = h + R \exp (i\psi)\) with \(-\pi / 2 \leq \psi \leq \pi / 2.\)

If \(r < \exp \left[ -\pi \tan \psi - \delta \sec \psi \right]\), then

\[
\int \frac{a(\omega)\omega}{\exp (2\pi i\omega) - 1} \, d\omega
\]

converges uniformly to zero as \(R\) becomes infinite. Let \(z = r \exp (i\pi)\) and \(\omega = h + R \exp (i\psi)\) with \(-\pi / 2 \leq \psi \leq \pi / 2.\)

If \(r < \exp \left[ -\pi \tan \psi - \delta \sec \psi \right]\), then

\[
\int \frac{a(\omega)\omega}{\exp (2\pi i\omega) - 1} \, d\omega
\]

converges uniformly to zero as \(R\) becomes infinite. Let \(z = r \exp (i\pi)\) and \(\omega = h + R \exp (i\psi)\) with \(-\pi / 2 \leq \psi \leq \pi / 2.\)

If \(r < \exp \left[ -\pi \tan \psi - \delta \sec \psi \right]\), then

\[
\int \frac{a(\omega)\omega}{\exp (2\pi i\omega) - 1} \, d\omega
\]

converges uniformly to zero as \(R\) becomes infinite. Let \(z = r \exp (i\pi)\) and \(\omega = h + R \exp (i\psi)\) with \(-\pi / 2 \leq \psi \leq \pi / 2.\)

If \(r < \exp \left[ -\pi \tan \psi - \delta \sec \psi \right]\), then

\[
\int \frac{a(\omega)\omega}{\exp (2\pi i\omega) - 1} \, d\omega
\]

converges uniformly to zero as \(R\) becomes infinite. Let \(z = r \exp (i\pi)\) and \(\omega = h + R \exp (i\psi)\) with \(-\pi / 2 \leq \psi \leq \pi / 2.\)

If \(r < \exp \left[ -\pi \tan \psi - \delta \sec \psi \right]\), then

\[
\int \frac{a(\omega)\omega}{\exp (2\pi i\omega) - 1} \, d\omega
\]

converges uniformly to zero as \(R\) becomes infinite. Let \(z = r \exp (i\pi)\) and \(\omega = h + R \exp (i\psi)\) with \(-\pi / 2 \leq \psi \leq \pi / 2.\)

If \(r < \exp \left[ -\pi \tan \psi - \delta \sec \psi \right]\), then

\[
\int \frac{a(\omega)\omega}{\exp (2\pi i\omega) - 1} \, d\omega
\]
Given an $\epsilon > 0$ we can choose an $R_1$ such that for $R > R_0$, $R_1$ and $0 \leq r \leq \exp \left[-\delta - \rho\right]$, $p > 0$, the quantity on the right of (27) is less than $\epsilon$. That is, (24) converges uniformly to zero as $R$ becomes infinite.

Let $\pi/2 \leq \psi \leq u < \pi$. If $r > \exp \left[\pi \tan \psi - \delta \sec \psi\right]$ then

\[ \text{(28)} \quad b_1(r, \psi) = \log r^{-1} \cos \psi + \pi \sin \psi - \delta > 0. \]

Let $-\pi < v \leq \psi < -\pi/2$. If $r > \exp \left[-\pi \tan \psi - \delta \sec \psi\right]$ then

\[ \text{(29)} \quad b_2(r, \psi) = \log r^{-1} \cos \psi + \pi \sin \psi - \delta > 0. \]

Denote, for a given $q > 0$ but otherwise arbitrarily small, by $F$ the smaller of the numbers $\pi \sin u - q$ and $\pi \sin v - q$. Let $\delta \leq F$. The maximum for $\pi/2 \leq \psi \leq u < \pi$ of $\pi \tan \psi - \delta \sec \psi$ is $\pi \tan \psi - F \sec \psi$. The maximum for $-\pi < v \leq \psi \leq -\pi/2$ of $-\pi \tan \psi - \delta \sec \psi$ is $\pi \tan v - F \sec v$. Denote by $E$ the larger of $\pi \tan u - F \sec u + t$ and $\pi \tan v - F \sec v + t$ where $t > 0$ but sufficiently small that $E < 0$. If $r > \exp (E)$ then (28) and (29) hold. Suppose now in addition that $u$ and $v$ have the property that

\[ \text{(30)} \quad \cos u > -\frac{\pi \sin u - F}{\delta + \rho + t} \]

and

\[ \text{(31)} \quad \cos v > -\frac{\pi \sin v - F}{\delta + \rho + t}. \]

Here $\rho$ and $t$ are positive but otherwise may be chosen arbitrarily small. Let $z = r \exp (i\pi)$ with

\[ \text{(32)} \quad \exp E \leq r \leq \exp (-\delta - \rho). \]

That there is an interval consisting of more than one point satisfying (32) follows from the restrictions (30) and (31) placed on $u$ and $v$. For if $u$ and $v$ satisfy (30) and (31) then $E < -\delta - \rho$. Denote by $b_1'' > 0$ the minimum assumed by $b_1(r, \psi)$ on $\pi/2 \leq \psi \leq u < \pi$ and by $b_2'' > 0$ the minimum assumed by $b_2(r, \psi)$ on $-\pi < v \leq \psi \leq -\pi/2$ where $r$ satisfies (32). Denote by $b'' > 0$ the smaller of $b_1''$ and $b_2''$. Let $b'' > 0$ be the smaller of $b_0$ and $b''$. Then by (6), (7) and (23) if $-\pi < v \leq \psi \leq u < \pi$, $\delta \leq F$ and $\exp E \leq r \leq \exp (-\delta - \rho)$, where $u$ and $v$ satisfy (30) and (31) we have
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\[ \int_\alpha \frac{a(\omega)z^{\omega}}{\exp(2\pi i\omega) - 1} \, d\omega \leq R^2 T_2(B, D) \int_\omega \exp(-b^*R) \, d\psi \leq R^2 T_2(B, D) \exp(-b^*(u - v)). \]

Given an \( \epsilon > 0 \) there exists an \( R_1 \) such that for \( R > R_0, R_1 \) and \( \exp E \leq r \leq \exp(-\delta - p) \) the quantity on the right of (33) is less than \( \epsilon \). Hence as \( R \) becomes infinite we again obtain (15) since (24) converges uniformly to zero.

It is now possible to consider again the four cases of §2; however, for brevity we shall consider only those corresponding to A and D. Let us denote these by A' and D'.

\[ A': \ 0 < \psi_1 < \pi/2, \ -\pi/2 < \psi_2 < 0, \]
\[ D': \ \pi/2 < \psi_1 \leq u, \ v \leq \psi_2 < -\pi/2. \]

Case A'. Suppose that \( \delta \leq \pi - d, \ d > 0 \). It will be shown that \( J_{\psi_1}(z) + J_{\psi_2}(z) \) is regular for \( z \) in a domain which includes all or part of the segment \( -\exp[-\delta - p] \leq \varepsilon < 0 \) in its interior. By a method similar to that employed in §2 we could show that for \( J_{\psi_1}(z) \) to converge for a fixed \( z = r \exp(i\theta) \) it is sufficient that

\[ 0 < r < \exp[\theta \tan \psi_1 - \delta \sec \psi_1], \ 0 \leq \theta < 2\pi. \]

It follows then that \( J_{\psi_1}(z) \) will converge uniformly for \( z \) in any closed domain contained in the domain defined by (34). Hence by reasoning similar to that employed in §2, it is easily seen that \( J_{\psi_1}(z) \) is regular for \( z \) in any closed domain contained in (34).

Consider \( J_{\psi_1}(z) \). It is easily shown that it converges for a fixed \( z = r \exp(i\theta) \) contained in the domain defined by

\[ 0 < r < \exp[(\theta - 2\pi) \tan \psi_2 - \delta \sec \psi_2], \ 0 \leq \theta < 2\pi. \]

If then \( z = r \exp(i\theta) \) is in any closed domain contained in the domain defined by (35), \( J_{\psi_2}(z) \) converges uniformly and hence represents a regular function. The function \( J_{\psi_1}(z) + J_{\psi_2}(z) \) will therefore be regular in any closed domain contained in the domain common to (34) and (35). We have seen that if \( z = r \exp(i\pi) \) with \( \exp E \leq r \leq \exp[-\delta - p] \), \( p > 0 \), that

\[ \sum_{n=0}^{\infty} a(n)z^n = J_{\psi_1}(z) + J_{\psi_2}(z). \]

But \( J_{\psi_1}(z) + J_{\psi_2}(z) \) has been shown to be regular in a domain which includes all or part of the interval \( \exp E \leq r \leq \exp[-\delta - p] \) in its interior. Therefore \( J_{\psi_1}(z) + J_{\psi_2}(z) + \sum_{n=0}^{\infty} a(n)z^n \) provides the analytic
continuation of \( f(z) \) to any closed domain contained in the domain common to (34) and (35). Hence in this case \( f(z) \) will be regular in a domain bounded by two spirals as indicated in Figure 4.

Consider case D'. Suppose \( \delta \leq F \) and that \( v \) and \( u \) satisfy (30) and (31). Then for \( E \leq r \leq \exp (-\delta - p) \) it follows from (33) that (15) holds. It is easily seen that \( J_{\psi_1}(z) \) will converge for a fixed \( z = r \exp (i\theta) \) if \( \log r^{-1} \cos \psi_1 + \theta \sin \psi_1 - \delta > 0 \). This will be the case if

\[
(36) \quad r > \exp [\theta \tan \psi_1 - \delta \sec \psi_1], \quad 0 \leq \theta < 2\pi.
\]

It is evident that \( J_{\psi_1}(z) \) will converge uniformly for \( z = r \exp (i\theta) \) in any closed bounded domain contained in the domain defined by (36). In order that \( J_{\psi_2}(z) \) converge for a fixed \( z = r \exp (i\theta) \) it is sufficient that \( \log r^{-1} \cos \psi_2 + (\theta - 2\pi) \sin \psi_2 - \delta > 0 \). That is,

\[
(37) \quad r > \exp [\{(\theta - 2\pi) \tan \psi_2 - \delta \sec \psi_2\}], \quad 0 \leq \theta < 2\pi.
\]
Then $J_{\psi_1}(z)$ will converge uniformly for $z = re^{i\theta}$ in any closed bounded domain contained in the domain defined by (37). Hence

$$J_{\psi_1}(z) + J_{\psi_2}(z)$$

will converge uniformly for $z = re^{i\theta}$ in any closed bounded domain contained in the domain common to (36) and (37). It is evident that the region common to (36) and (37) contains the interval $-\exp(-\delta - \rho) \leq z \leq -\exp E$ in its interior. For $E$ is by definition the larger of $\pi \tan u - F \sec u + t$ and $\pi \tan v - F \sec v + t$, $t > 0$, and if we set $\theta = \pi$ in (36) and (37) it is clear that both exponents are smaller than $E$. Therefore $J_{\psi_1}(z) + J_{\psi_2}(z) + \sum_{n=1}^{\infty} a(n)z^n$ provides the analytic continuation of $f(z)$ to any closed bounded domain contained in the domain common to (36) and (37). For the case $\psi_1 = \psi_2$ this will be such a domain as indicated in Figure 5.
Hence we have the following theorem.

**Theorem 2.** Let \( f(z) = \sum_{n=0}^{\infty} a_n z^n \) with radius of convergence unity. Let the coefficients \( a_n \) be the values taken on by an analytic function \( a(z) \) at \( z = 0, 1, 2, \ldots \). Suppose \( a(z) \) is regular with the possible exception of an essential singularity at infinity in an angle with vertex \( h>0 \) (non-integral) on the real axis, including the axis of positive reals in its interior. Let the sides of this angle make angles \( \psi_1 \) and \( \psi_2 \) with the axis of positive reals. Then if, for \( z = h + R \exp (i\psi) \) in this angular opening, \( a(z) \) satisfies the inequality

\[
|a(h + R \exp (i\psi))| < \exp (\delta R), \quad R > R_0,
\]

where \( \delta \leq \pi - d \), \( d > 0 \), and

\[
A': \quad 0 < \psi_1 < \pi/2, \quad -\pi/2 < \psi_2 < 0,
\]

\( f(z) \) is regular in the domain common to

\[
r \leq \exp \left[ \theta \tan \psi_1 - \delta \sec \psi_1 \right] - \gamma, \quad \gamma > 0
\]

and

\[
r \leq \exp \left[ (\theta - 2\pi) \tan \psi_2 - \delta \sec \psi_2 \right] - \gamma, \quad \gamma > 0,
\]

for \( 0 \leq \theta < 2\pi \).

For a given \( q > 0 \) but otherwise arbitrarily small let \( F \) be the smaller of the numbers \( \pi \sin u - q \) and \( \pi \sin v - q \). Suppose \( u \) and \( v \) may be chosen such that for a given \( p \) and \( t \) positive but otherwise arbitrarily small

\[
\cos u > -\frac{\pi \sin u - F}{\delta + p + t}
\]

and

\[
\cos v > -\frac{\pi \sin v - F}{\delta + p + t}.
\]

Then if \( \delta \leq F \) and

\[
D': \quad \pi/2 < \psi_1 \leq u < \pi, \quad -\pi < v \leq \psi_2 < -\pi/2,
\]

\( f(z) \) is regular in any bounded domain common to

\[
r \geq \exp \left[ \theta \tan \psi_1 - \delta \sec \psi_1 \right] + \gamma, \quad \gamma > 0,
\]

and

\[
r \geq \exp \left[ (\theta - 2\pi) \tan \psi_2 - \delta \sec \psi_2 \right] + \gamma, \quad \gamma > 0,
\]

where \( 0 \leq \theta < 2\pi \).
4. Conclusions. We note first that if, in Theorem 1, \( \psi_1 > 0 \) and \( \psi_2 < 0 \) but otherwise arbitrarily small, that \( f(z) \) has \( z = 1 \) as its only singularity on the circle of convergence.

In part D of Theorem 1 if both \( \psi_1 \) and \( \psi_2 \) are greater than 90° in magnitude, that is, the sector of regularity is greater than 180°, we have the rather remarkable result that \( z = 1 \) is the only singularity of \( f(z) \) in the finite plane. Thus for example the function defined by the series

\[
\sum_{n=0}^{\infty} \frac{z^n}{(n + \alpha)^\beta},
\]

where \( \beta \) is an integer and \( \alpha \) is not equal to zero or a negative integer but otherwise arbitrary, has the point \( z = 1 \) as its only singularity in the finite plane.

It is of course clear that we may use the results obtained in a different manner, that is, if \( f(z) \) has a singular point on the circle of convergence other than \( z = 1 \) then \( a(z) \) cannot be analytic in an angular opening including the axis of positive reals in its interior with at most a pole of finite order at infinity.

If the inequality for \( a(z) \) in Theorem 2 is satisfied for every \( \delta > 0 \) however small, then under the condition of case D' of Theorem 2, \( z = 1 \) is the only singularity in the finite plane. This result is analogous to the following theorem due to Faber [1].

**Theorem (Faber).** If \( g(z) \) is an integral function such that \( |g(re^{i\theta})| < e^r \) for an arbitrary positive \( e \) and \( r > r' \), the function \( f(z) \) defined by \( \sum_{n=0}^{\infty} (n + \alpha)^{\beta} \) and its analytic continuation has the point 1 as its only singular point.

We observe now that the bounding curves

\[
r < \exp \left[ \theta \tan \psi_1 - \delta \sec \psi_1 \right], \quad 0 \leq \theta < 2\pi,
\]

and

\[
r < \exp \left[ (\theta - 2\pi) \tan \psi_2 - \delta \sec \psi_2 \right], \quad 0 \leq \theta < 2\pi,
\]

of (34) and (35) cut the unit circle at the points \( \exp[i\delta \csc \psi_1] \) and \( \exp[i(2\pi + \delta \csc \psi_2)] \). If now, in addition to the requirements of part A of Theorem 2, \( \psi_1, \psi_2 \) and \( \delta \) satisfy the inequality

\[
2\pi > \delta(\csc \psi_1 - \csc \psi_2)
\]

it is easily seen that the region common to (34) and (35) will extend beyond the unit circle. We then have the following theorem.

---

* The author is indebted to the referee for pointing out the analogy.
THEOREM 3. If the conditions of Theorem 2 part A are satisfied and if in addition the quantities \( \psi_1, \psi_2 \) and \( \delta \) satisfy the inequality
\[
2\pi > \delta (\csc \psi_1 - \csc \psi_2)
\]
then the circle of convergence is not a cut for the function.

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A NOTE ON THE HILBERT TRANSFORM

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The Hilbert transform of \( f(t), -\infty < t < \infty \), is \( 1/\pi \) times the Cauchy principal value
\[
\tilde{f}(x) = P \int_{-\infty}^{\infty} \frac{f(t)}{t - x} \, dt = \lim_{\delta \to 0^+} \int_{\delta}^\infty \frac{f(x + t) - f(x - t)}{t} \, dt.
\]

If \( f(t) \in L_p, p > 1 \), then \( \tilde{f}(x) \in L_p \), and a considerable literature is devoted to studying the relationship of such pairs of "conjugate" functions to the theory of functions analytic in a half-plane. More to the point of the present note is a series of papers studying the Hilbert transform along strictly real variable lines ([2, 3]; further bibliography in [2]).

Much less is known about \( \tilde{f}(x) \) when \( f(t) \in L_1 \). Plessner found by applying complex variable methods to the theory of Fourier series that if \( f(t) \in L_1 \) then \( \tilde{f}(x) \) exists almost everywhere (see [1, p. 145]). Besicovitch [4] proved Plessner's result using only the theory of sets, starting from his own previous real variable investigation of the \( L_2 \) transform case. S. Pollard [5] showed how Besicovitch's proof could be extended to prove the existence a.e. of the principal value of the Stieltjes integral
\[
\tilde{f}(x) = P \int_{-\infty}^{\infty} \frac{dF(t)}{t - x},
\]

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1 Numbers in brackets refer to the bibliography at the end of the paper.