Theorem 3. If the conditions of Theorem 2 part A are satisfied and if in addition the quantities \( \psi_1, \psi_2 \) and \( \delta \) satisfy the inequality

\[
2\pi > \delta (\csc \psi_1 - \csc \psi_2)
\]

then the circle of convergence is not a cut for the function.

Bibliography


Lehigh University

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A NOTE ON THE HILBERT TRANSFORM

LYNN H. LOOMIS

The Hilbert transform of \( f(t) \), \( -\infty < t < \infty \), is \( 1/\pi \) times the Cauchy principal value

\[
\tilde{f}(x) = P \int_{-\infty}^{\infty} \frac{f(t)}{t - x} \, dt = \lim_{\delta \to 0^+} \int_{\delta}^{\infty} \frac{f(x + t) - f(x - t)}{t} \, dt.
\]

If \( f(t) \in L_p, \ p > 1 \), then \( \tilde{f}(x) \in L_p \), and a considerable literature is devoted to studying the relationship of such pairs of "conjugate" functions to the theory of functions analytic in a half-plane. More to the point of the present note is a series of papers studying the Hilbert transform along strictly real variable lines ([2, 3]; further bibliography in [2]).

Much less is known about \( \tilde{f}(x) \) when \( f(t) \in L_1 \). Plessner found by applying complex variable methods to the theory of Fourier series that if \( f(t) \in L_1 \) then \( \tilde{f}(x) \) exists almost everywhere (see [1, p. 145]). Besicovitch [4] proved Plessner's result using only the theory of sets, starting from his own previous real variable investigation of the \( L_2 \) transform case. S. Pollard [5] showed how Besicovitch's proof could be extended to prove the existence a.e. of the principal value of the Stieltjes integral

\[
\tilde{f}(x) = P \int_{-\infty}^{\infty} \frac{dF(t)}{t - x},
\]

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1 Numbers in brackets refer to the bibliography at the end of the paper.
where $F(t)$ is continuous and of bounded variation over $(-\infty, \infty)$. In general $\tilde{f}(x)$ is not summable, but Kolmogoroff [6] found, using a contradiction argument, that there exists a constant $A$ such that the set where $\tilde{f}(x) > M > 0$ has measure at most $A \|f\| / M$, where $\|f\| = \int_{-\infty}^{\infty} |f(t)| \, dt$. Titchmarsh [7] was able to refine Besicovitch's existence proof so that it implied this bound, with a numerical value for $A$.

The present note contains a new direct real-variable proof of the Plessner existence theorem and the Kolmogoroff bound. In fact, this bound in a sense is the central tool for the existence proof, a device which allows for the first time the $L_1$ results to be obtained without recourse to the $L_2$ transform theory.

**Lemma 1.** If $c_i > 0$ and

$$g(x) = \sum_{i=1}^{n} \frac{c_i}{x - a_i},$$

then the set of points where $g(x) > M$ ($M > 0$) consists of $n$ intervals whose total length is precisely $(\sum c_i) / M$. The set where $g(x) < -M$ has the same length.

Since $g(a_i -) = -\infty$, $g(a_i +) = \infty$ and $g'(x) < 0$ for all $x$, there are precisely $n$ points $m_i$ such that $g(m_i) = M$, and $a_i < m_i < a_{i+1}$, $i = 1, \ldots, n - 1$, $a_n < m_n$. The set where $g(x) > M$ thus consists of the intervals $(a_i, m_i)$ and has the total length

$$\sum_{i=1}^{n} (m_i - a_i) = \sum_{i=1}^{n} m_i - \sum_{i=1}^{n} a_i. \quad (1)$$

But the numbers $m_i$ are the roots of the equation

$$\sum_{i=1}^{n} \frac{c_i}{x - a_i} = M,$$

whose cross-multiplied form is

$$\sum_{i=1}^{n} c_i \left[ \prod_{j \neq i} (x - a_j) \right] = M \prod_{i=1}^{n} (x - a_i),$$

or

$$M x^n - \left[ M \sum_{i=1}^{n} a_i + \sum_{i=1}^{n} c_i x^{n-1} + \cdots \right] = 0,$$

so that

$$\sum_{i=1}^{n} m_i = \sum_{i=1}^{n} a_i + \frac{1}{M} \sum_{i=1}^{n} c_i. \quad (2)$$
The first part of the lemma follows from (1) and (2); the proof for $g(x) < -M$ is almost identical.

**Lemma 2.** Let $F(t)$ be increasing over $(-\infty, \infty)$ with finite total variation $V(F)$. If $(x_i - \delta_i, x_i + \delta_i), j = 1, \cdots, n,$ are disjoint intervals such that

$$\int_{x_i - \delta_i}^{x_i + \delta_i} dF(t) > M > 0,$$

then $\sum \delta_i \leq 4V(F)/M$. The same inequality is implied if the integral is less than $-M, j = 1, \cdots, n.$

Let $t_i, i = 1, \cdots, N$ be a finite subdivision including the points $x_i - \delta_i, x_i, x_i + \delta_i$ for $j = 1, \cdots, n,$ and such that the approximating Riemann sums for (3), with the integrand evaluated at the left-hand end points, remain greater than $M$. Thus, if $\Delta_i = F(t_{i+1}) - F(t_i),$

$$\sum_{i \in I_j} \frac{\Delta_i}{t_i - y} > M$$

for $y = x_i$, where the set $I_j$ of omitted indices is defined by

$$\bigcup_{i \in I_j} (t_i, t_{i+1}) = (x_i - \delta_i, x_i + \delta_i).$$

Since the left member of (4) is an increasing function of $y$ for $x_i - \delta_i < y < x_i + \delta_i$, the inequality (4) holds for $x_i \leq y < x_i + \delta_i$. For every such $y$ one of the following inequalities is therefore satisfied:

$$\sum_{i=1}^{N-1} \frac{\Delta_i}{t_i - y} > \frac{M}{2}, \quad \sum_{i \in I_j} \frac{\Delta_i}{t_i - y} < -\frac{M}{2}.$$

Applying Lemma 1 and summing over $j$, we have

$$\sum \delta_i \leq \sum_{i=1}^{N-1} \frac{2\Delta_i}{M} + \sum_{j=1}^{n} \sum_{i \in I_j} \frac{2\Delta_i}{M} \leq \frac{4}{M} \sum_{i=1}^{N-1} \Delta_i \leq \frac{4}{M} V(F).$$

To prove the second part of the lemma we only need to observe that the integral in (3) is less than $-M$ if and only if after replacing $F(t)$ by $-F(-t)$ and $x_i$ by $-x_i$ it is greater than $M$.

**Corollary.** If $F(t)$ is of bounded variation in Lemma 2 then $\sum \delta_i \leq 8V(F)/M$.

This follows at once upon applying the lemma to the increasing and decreasing parts, $F_1$ and $F_2,$ of $F$, using $V(F) = V(F_1) + V(F_2)$.

Preliminary to the theorem we remark that if $f(t)$ has the value 1 in
(a, b) and 0 elsewhere, then its Hilbert transform exists except at the two points a and b, and has the value \( \log \left| \frac{x-b}{x-a} \right| \). In particular the Hilbert transform of any step function exists except at a finite number of points.

**Theorem.** Let \( F(t) \) be of bounded variation over \( (-\infty, \infty) \). Then its Hilbert-Stieltjes transform

\[
\tilde{f}(x) = P \int_{-\infty}^{\infty} \frac{dF(t)}{t-x}
\]

exists almost everywhere, and, for every positive \( M \), the set where \( \tilde{f}(x) > M \) has measure at most \( 16V(F)/M \), as does the set where \( \tilde{f}(x) < M \).

We first prove the existence of \( \tilde{f}(x) \). It is sufficient to show that, given \( \epsilon \), for every \( x \) except in a set of measure less than \( \epsilon \)

\[
\left| \int_{x-\delta}^{x-\delta'} + \int_{x+\delta}^{x+\delta'} \frac{dF(t)}{t-x} \right| \leq \epsilon
\]

for all sufficiently small \( \delta \) and \( \delta' \). Now the absolutely continuous part of \( F \) can be approximated to within \( \epsilon' \) by the integral \( F_1 \) of a step function \( h, F_1(t) = \int_{x=0}^{x} h(t) dt \), and the singular part of \( F \) can be approximated to within \( \epsilon' \) by a singular function \( F_2 \) whose variation is confined to a closed set of measure 0, that is, which is constant on the intervals of an open set \( M \) whose complement has measure zero. Thus, taking \( \epsilon' = \epsilon^2/192 \), we have \( F = F_1 + F_2 + F_3 \), where \( V(F_3) < \epsilon^2/96 \). Let \( E_\epsilon \) be the set of \( x \) for which the inequality

\[
\left| \int_{x-\delta}^{x-\delta'} + \int_{x+\delta}^{x+\delta'} \frac{dF_3(t)}{t-x} \right| \leq \frac{\epsilon}{3}
\]

fails to hold for arbitrarily small \( \delta \) and \( \delta' \) (\( \delta' < \delta \)). Then for every \( x \) in \( E_\epsilon \)

\[
\left| \int_{x-\Delta}^{x-\Delta} + \int_{x+\Delta}^{x+\Delta} \frac{dF_3(t)}{t-x} \right| > \frac{\epsilon}{6}
\]

for arbitrarily small \( \Delta \). By Vitali’s theorem a disjoint sequence of intervals \( (x_i - \Delta_i, x_i + \Delta_i) \) satisfying \( (7) \) can be chosen so as to cover \( E_\epsilon \) except for a set of measure 0. Then by Lemma 2, corollary, \( m(E_\epsilon) \leq 2 \sum \Delta_i \leq 2 \cdot 8 V(F_3) \cdot 6/\epsilon < \epsilon \).

Since \( F_1(t) \) is the integral of a step function, its Hilbert-Stieltjes transform (the Hilbert transform of \( h(t) \)) exists except at a finite number of points, which we add to \( E_\epsilon \). Since \( F_2(t) \) is constant on the intervals of \( M \), its Hilbert-Stieltjes transform obviously exists except
on the complement of $M$, which we add to $E_*$. Thus if $x$ is not in the enlarged $E_*$, there is a $\Delta$ such that, for all $\delta$ and $\delta'$ less than $\Delta$, (6) holds for $F_1$, $F_2$ and $F_3$, and hence (5) holds, as was to be proved.

The second part of the theorem follows immediately from Lemma 2, corollary, where the intervals $(x_i - \delta_i, x_i + \delta_i)$ are chosen by Vitali's theorem to cover almost entirely the set where $\bar{f}(x) > M$ ($\bar{f}(x) < -M$), so that the measure of this set is not greater than $2\sum \delta_i \leq 16V(F)/M$.

**Corollary.** If $0 < p < 1$ and $p + q > 1$, then $|\bar{f}(x)|^{p}/(1 + |x|)^q \in L_1$.

This follows immediately from the fact that the decreasing function on $(0, \infty)$ which is equimeasurable with $|\bar{f}(x)|$ is dominated by $K/x$.

In case $F(t)$ is singular and increasing, it can be shown with little difficulty that the constant 16 can be replaced by 1, and this is best possible since $1/x$ itself is the Hilbert-Stieltjes transform of the function $F(t)$ which is 1 when $t < 0$ and 0 when $t \geq 0$. This is probably the correct value of the constant in the general case.

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