A PARTICULAR GENERALIZED LAPLACIAN

MAXWELL O. READE

In a preceding paper, the author developed a generalized Laplacian for functions having subharmonic logarithms [2]. The purpose of this note is to indicate how generalized Laplacians may be used to weaken differentiability requirements; in particular, the generalized Laplacian

\[ \Delta^* f(x, y) \equiv \limsup_{\rho \to 0} \frac{4}{\rho^2} \left[ L(f; x, y; \rho) - f(x, y) \right] \]

is used to weaken differentiability requirements in certain theorems due to Kierst and Saks [4] and the author [3].

The definitions and notation used in [2] will be used here. In addition, use will be made of the following known result.

**Theorem A [1].** If \( f(x, y) \) is continuous in a domain \( G \), then a necessary and sufficient condition that \( f(x, y) \) be subharmonic in \( G \) is that

\[ \Delta^* f(x, y) \geq 0 \]

hold throughout \( G \).

A slightly more general version of a theorem due to Kierst and Saks is the following one.

**Theorem 1.** Let \( F(t) \) have a continuous second derivative, with \( F'(t) > 0 \), for \( -\infty < t < \infty \). If \( f(x, y) \) has continuous partial derivatives of the first order in a domain \( G \), and if \( F[ax + \beta y + f(x, y)] \) is subharmonic in \( G \) for every choice of the real constants \( \alpha, \beta \), then \( f(x, y) \) is subharmonic in \( G \).

**Proof.** Let \((x_0, y_0)\) be a fixed, arbitrary point of \( G \). Then after expanding \( F(t) \) and \( f(x, y) \) in Taylor series about \( t_0 = ax_0 + \beta y_0 + v(x_0, y_0) \) and \((x_0, y_0)\), respectively, one obtains

\[ L(\phi_{\alpha, \beta}; x_0, y_0; \rho) - \phi_{\alpha, \beta}(x_0, y_0) = F'(t_0) \left[ L(f; x_0, y_0; \rho) - f(x_0, y_0) \right] \]

\[ + \frac{\rho^2 F''(t_0)}{4} \left[ (\alpha + f_0)^2 + (\beta + f_0)^2 \right] + o(\rho^2), \]

where

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1 Numbers in brackets refer to the bibliography at the end of the paper.
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\[ \phi_{\alpha, \beta}(x, y) = F[\alpha x + \beta y + f(x, y)], \]

\[ f_x = \left( \frac{\partial f}{\partial x} \right)_{(x_0, y_0)}, \quad f_y = \left( \frac{\partial f}{\partial y} \right)_{(x_0, y_0)}, \]

and \( o(\rho^2) \) is a quantity (not always the same quantity) such that

\[ \lim_{\rho \to 0} \frac{o(\rho^2)}{\rho^2} = 0. \]

Now set \( \alpha = -f_x, \beta = -f_y \). Since \( \phi_{\alpha, \beta}(x, y) \) is subharmonic in \( G \) for all choices of the real constants \( \alpha, \beta \), it follows from (1), (3), (4) and Theorem A that (2) holds at \( (x_0, y_0) \). But \( (x_0, y_0) \) was an arbitrary point of \( G \), so that (2) holds throughout \( G \); therefore, by Theorem A, \( f(x, y) \) is subharmonic in \( G \). This completes the proof.

In a similar manner one may prove the following more general version of a theorem due to the author [3].

\textbf{Theorem 2.} Let \( F(t) \) and \( f(x, y) \) have the properties noted in Theorem 1. If the function \( F\{ \log((x - \alpha)^2 + (y - \beta)^2) + f(x, y) \} \) is subharmonic in \( G \) for every choice of the real constants \( \alpha, \beta \), then \( f(x, y) \) is subharmonic in \( G \).

The same technique may be applied to other results [1, 3] to obtain slightly more general theorems. However, it would be desirable to remove all conditions of differentiability (on \( f(x, y) \))—which the usual averaging process does not appear to do.

Other generalized Laplacians may be used to obtain results similar to those above; for example, either

\[ \lim_{\rho \to 0} \sup_{\rho^2} \frac{8}{\rho^2} [A(f; x, y; \rho) - f(x, y)], \]

or

\[ \lim_{\rho \to 0} \sup_{\rho^2} \frac{8}{\rho^2} [L(f; x, y; \rho) - A(f; x, y; \rho)] \]

may be used.

\textbf{Bibliography}


Purdue University