ON k-TO-1 TRANSFORMATIONS

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The following results are extensions of certain of the theorems of O. G. Harrold (Exactly (k, 1) transformations on connected linear graphs, Amer. J. Math. vol. 62 (1940) pp. 823–834).

Let X and Y be compact Hausdorff spaces and let f be a continuous transformation of X onto Y. Let k be a positive integer and let μE denote the cardinal of the set E. We say that f is at most k-to-1 (or exactly k-to-1) in case y ∈ Y implies μf⁻¹(y) ≤ k (or μf⁻¹(y) = k). Let o(x) denote the order of the point x. That is to say, o(x) is the smallest integer m such that μ bdy U = m for an arbitrarily small open neighborhood U of x, if such exists; otherwise o(x) is ∞.

**THEOREM 1.** If f is at most k-to-1 and if the inverse points of y ∈ Y are x₁, · · · , xₙ, then \( \sum_{i=1}^{n} o(x_i) \leq k \cdot o(y). \)

**PROOF.** We may suppose o(y) is finite. Let U₁, · · · , Uₙ be neighborhoods (open neighborhoods) of x₁, · · · , xₙ whose closures are pairwise disjoint. There exists a neighborhood W of y such that μ bdy W = o(y) and f⁻¹(W) ⊆ ∪ᵢ Uᵢ. Define Vᵢ = Uᵢ ∩ f⁻¹(W). It follows that k · o(y) = k · μ bdy W ≥ μf⁻¹(bdy W) ≥ μ bdy f⁻¹(W) = μ ∪ᵢ bdy Vᵢ = \( \sum_{i=1}^{n} \mu \cdot bdy Vᵢ \). We conclude that each o(xᵢ) is finite. By taking the Uᵢ sufficiently small, μ bdy Vᵢ ≥ o(xᵢ). The conclusion follows.

**COROLLARY 1.** If X and Y are continua and if f is exactly k-to-1, then each inverse point of an end point of Y is an end point of X.

Let P denote the property of being a continuum in X on which f is exactly k-to-1.

**THEOREM 2.** If X has property P irreducibly, then Y has no end point; if moreover k = 2, then Y has no cut point.

**PROOF.** We prove the first statement. Suppose Y has an end point y. Write f⁻¹(y) = ∪ᵢ Uᵢ. Let U₁, · · · , Uₖ be neighborhoods of x₁, · · · , xₖ whose closures are pairwise disjoint. There exists a neighborhood W of y such that μ bdy W = o(y) = 1 and f⁻¹(W) ⊆ ∪ᵢ Uᵢ. Define Vᵢ = Uᵢ ∩ f⁻¹(W). As in the proof of Theorem 1 it follows that k = k · o(y) ≥ \( \sum_{i=1}^{k} \mu \cdot bdy Vᵢ \). Hence, each bdy Vᵢ consists of a single point and it follows easily that \( X - ∪ᵢ Vᵢ = X - f⁻¹(W) \) is a proper

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subcontinuum of $X$ on which $f$ is exactly $k$-to-1. This is a contradiction.

We prove the second statement. Suppose $y$ is a cut point of $Y$ and let $Y - y = Y_1 \cup Y_2$ be a separation. Then, $X - f^{-1}(y) = f^{-1}(Y_1) \cup f^{-1}(Y_2)$ is a separation and since also $f^{-1}(y)$ consists of only two points, at least one of the sets $f^{-1}(Y_i) \cup f^{-1}(y)$ ($i = 1, 2$) is a continuum. This contradicts the hypothesis of irreducibility.

**Corollary 2.** No dendrite is a continuous exactly $k$-to-1, $k > 1$, image of a continuum.

**Proof.** Suppose $f(X) = Y$ is exactly $k$-to-1, $k > 1$, where $X$ is a continuum and $Y$ is a dendrite. By use of Zorn's lemma it may readily be seen that there exists a subcontinuum $X_0$ of $X$ which has property P irreducibly. The nondegenerate continuum $f(X_0)$ is a dendrite and hence has an end point. This is impossible by Theorem 2.

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