ON THE INTERIOR OF THE CONVEX HULL OF
A EUCLIDEAN SET

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In this note we shall prove for each positive integer \( n \) the following theorem \( \Delta_n \) concerning convex sets in an \( n \)-dimensional euclidean space.

**Theorem \( \Delta_n \).** Any point interior to the convex hull of a set \( E \) in an \( n \)-dimensional euclidean space is interior to the convex hull of some subset of \( E \) containing at most 2\( n \) points.

This theorem is similar to the well known result that any point in the convex hull of a set \( E \) in an \( n \)-dimensional euclidean space lies in the convex hull of some subset of \( E \) containing at most \( n+1 \) points \([1, 2]\).\(^1\) In these theorems the set \( E \) is an arbitrary set in the space. The convex hull of \( E \), denoted by \( H(E) \), is the set product of all convex sets in the space which contain \( E \).

A euclidean subspace of dimension \( n-1 \) in an \( n \)-dimensional euclidean space will be called a plane. Every plane in an \( n \)-dimensional euclidean space separates its complement in the space into two convex open sets, called open half-spaces, whose closures are convex closed sets, called closed half-spaces. If each of the two open half-spaces bounded by a plane \( L \) intersects a given set \( E \), then \( L \) is said to be a separating plane of \( E \); otherwise \( L \) is said to be a nonseparating plane of \( E \).

In order to prove our sequence of theorems we shall make use of the following result: A point \( i \) is interior to the convex hull of a set \( E \) in an \( n \)-dimensional euclidean space if and only if every plane through \( i \) is a separating plane of \( E \) \([1]\).

We prove our sequence of theorems by induction. The proof of Theorem \( \Delta_1 \) is trivial and will be omitted. Now suppose that Theorem \( \Delta_{n-1} \) is true for an integer \( n > 1 \). We shall show that Theorem \( \Delta_n \) is also true. To this end let \( i \) be a point interior to the convex hull of a set \( E \) in an \( n \)-dimensional euclidean space. We are to demonstrate that \( i \) is interior to the convex hull of some subset \( P \) of \( E \) containing at most \( 2n \) points.

First we show that \( i \) is interior to the convex hull of some finite subset \( Q \) of \( E \). Since \( i \) is interior to \( H(E) \), it is interior to a simplex

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\(^1\) Numbers in brackets refer to the references cited at the end of the paper.

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lying in $H(E)$. Consider the $n+1$ vertices $q_k$ ($k=1, \ldots, n+1$) of such a simplex. The vertex $q_k$ lies in $H(E)$ and hence, according to the previously mentioned result of Carathéodory and Steinitz, lies in the convex hull of some subset $Q_k$ of $E$ containing at most $n+1$ points. The set $Q=\sum_k Q_k$ is then a finite subset of $E$ containing at most $(n+1)^2$ points. Evidently the convex hull of this set contains the simplex with vertices $q_k$ and hence contains the point $i$ in its interior.\footnote{That $i$ is interior to the convex hull of some finite subset of $E$ may also be proved by the Heine-Borel theorem. I am indebted to the referee for the above proof.}

Since $Q$ is finite, there exists a subset $P$ of $Q$ which contains the point $i$ in the interior of its convex hull and which is irreducible with respect to this property. Let $p$ be a definite point of $P$. Then $i$ is not an interior point of $H(P-p)$, so some plane $L$ through $i$ is a non-separating plane of $P-p$. Let $D$ be that one of the two open half-spaces bounded by $L$ which is disjoint with $P-p$ and let $D'$ be the other open half-space. Thus $P-p$ lies in the closed half-space $\overline{D'}$ complementary to $D$.

Since $i$ is an interior point of the convex hull of $P$, the open half-space $D$ contains a point of $P$. This point must be $p$, for $D$ contains no point of $P-p$. Similarly the open half-space $D'$ contains a point $p'$ of $P$. We shall use this point $p'$ later in the proof.

Consider an arbitrary point $x$ of the closed half-space $\overline{D'}$. Since $p$ lies in the complementary open half-space $D$, the line segment $H(p+x)$ intersects the boundary $L$ of $D$ in exactly one point which we denote by $\phi(x)$. Thus $\phi(x)$ is the projection of $x$ from $p$ onto $L$.

The projection $\phi$ is 1-1 over the subset $P-p$ of the closed half-space $\overline{D'}$. For suppose, to the contrary, that some two points $p_1$ and $p_2$ of $P-p$ project into the same point of $L$. The three points $p$, $p_1$, and $p_2$ are then collinear. Now $p$ does not lie between the other two points, else the open half-space $D$ containing $p$ would contain at least one of these other two points. We may then assume $p_1$ and $p_2$ to be so labeled that a linear order of the three points is $p$, $p_1$, $p_2$. Therefore

$$p_1 \subset H(p+p_2) \subset H(P-p),$$

so the sets $H(P-p_1)$ and $H(P)$ are identical. The point $i$ is then interior to $H(P-p_1)$ in contradiction to the irreducibility of $P$.

The projection of the convex hull of a set is the convex hull of the projection of that set, and the projection of an interior point of a convex set is an interior point of the projection of that set [3]. Therefore the point $\phi(i) = i$ is an interior point of the set $\phi(H(P-p)) = H(\phi(P-p))$ in the euclidean subspace $L$ of dimension $n-1$. Ac-
according to Theorem \( \Delta_{n-1} \) the point \( i \) is an interior point in \( L \) of the convex hull of some subset \( P_L \) of \( \phi(P - \rho) \) containing at most \( 2n - 2 \) points. Define

\[
P^* = \rho + P\phi^{-1}(P_L) + \rho'.
\]

Since the projection \( \phi \) is 1-1 over \( P - \rho \), the set \( P\phi^{-1}(P_L) \) is a subset of \( P \) containing at most \( 2n - 2 \) points. Therefore \( P^* \) is a subset of \( P \) containing at most \( 2n \) points.

We shall show that \( i \) is interior to \( H(P^*) \). First we notice that the coplanar set \( P_L \) lies in \( H(P^*) \). For, if \( x \) is an arbitrary point of \( P\phi^{-1}(P_L) \), then

\[
\phi(x) \subset H(\rho + x) \subset H(P^*),
\]

since both \( \rho \) and \( x \) lie in \( H(P^*) \). Now consider the pyramid \( H(\rho + P_L) \) whose apex \( \rho \) lies in \( D \) and whose base \( H(P_L) \) lies in \( L \). The point \( i \) is an interior point in \( L \) of the base \( H(P_L) \) of this pyramid, so some closed hemisphere \( A \) with center \( i \) and base on \( L \) lies in \( H(\rho + P_L) \).

Similarly, some closed hemisphere \( A' \) with center \( i \) and base on \( L \) lies in the pyramid \( H(\rho' + P_L) \). Evidently there exists a sphere \( I \) with center \( i \) such that \( I \subset A + A' \subset H(\rho + P_L) + H(\rho' + P_L) \subset H(P^*) \). The point \( i \) is then interior to the convex hull of the subset \( P^* \) of \( P \). From the irreducibility of \( P \) it follows that \( P^* = P \). Therefore \( P \) contains at most \( 2n \) points.

Thus for every integer \( n > 1 \), Theorem \( \Delta_{n-1} \) implies Theorem \( \Delta_n \). Since Theorem \( \Delta_1 \) is true, we conclude by induction that Theorem \( \Delta_n \) is true for each positive integer \( n \).

The following example shows that the number \( 2n \) in Theorem \( \Delta_n \) cannot be improved. Let \( i \) be the zero point of an \( n \)-dimensional vector space. Choose any \( n \) linearly independent and hence nonzero points in this space. Let \( E \) be the set consisting of these points and their vector negatives; \( E \) then contains \( 2n \) points. It is easy to show that the zero point \( i \) is interior to the convex hull of \( E \) but is not interior to the convex hull of any proper subset of \( E \).

**References**