A NOTE ON THE MINIMUM MODULUS OF A CLASS OF INTEGRAL FUNCTIONS

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A well known theorem due to Littlewood, Wiman, and Valiron states that for any integral function of order less than one-half,

$$\log m(r) > (\text{a positive constant}) \log M(r),$$
on a sequence of circles of indefinitely increasing radius. I consider in this note a class of integral functions which have this property and prove the following theorem.

**Theorem 1. Hypothesis:**

1. \((R_n)\) is any sequence of positive numbers such that \(R_1 > 1, R_n/R_{n-1} \geq \lambda > 1.\)
2. \((p_n)\) is any sequence of positive integers.
3. \(a_{11}, a_{12}, \ldots, a_{1p_1}, a_{21}, \ldots, a_{2p_2}, \ldots\) are a set of points such that \(0 < |a_{11}| \leq |a_{12}| \leq \cdots \) and such that a finite number \(a_{n1}, \ldots, a_{np_n}\) lie inside the ring \((R_n - R^a_n < |z| < R_n)\) where \(0 < \alpha < 1.\)
4. \(\mu_n\) is a sequence of positive integers such that \(\sum \frac{p_n}{\beta^{\mu_n}}\) is convergent, \(\beta\) being any constant greater than one.
5. The exponent of convergence of the points

$$a_n \exp \left(2\pi i\nu/\mu_n\right),$$

where \(r = 1, 2, \ldots, p_n; \nu = 0, 1, 2, \ldots, \mu_n - 1; n = 1, 2, 3, \ldots,\) is \(\rho\)
\((0 \leq \rho < \infty).\)
6. *Lower bound \(\mu_n \geq 1 + \rho.\)

**Conclusion:**

7. The canonical product

$$f(z) = \prod_{n=1}^{\infty} \prod_{\nu=1}^{p_n} \left(1 - \frac{z^{\mu_n}}{a_{n\nu}^{\beta^{\mu_n}}}\right)$$

formed with these points as zeros is of order \(\rho\); and the values of \(r = |z|\)
for which the inequality

$$m(r, f) > CM(r, f),$$

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2 It is possible to choose \(R_n, p_n,\) and so on, satisfying the conditions (1) to (6).
Example: \(R_n = 2^n; p_n = n^{2^n}; \mu_n = 2^n.\) Here \(\rho = 1.\)
where \( C = C(\lambda, \epsilon) > 0 \), is satisfied for a set of upper density greater than \( 1 - 1/\lambda - \epsilon \).

**Theorem 2.** If (1), (2), (3), (4), (5), and (6) hold and if \( \rho > 0 \) and if further\(^8\)

\[
\sum_{n=1}^{N} \frac{\mu_n p_n}{R_N^\rho} \to \infty \quad \text{with } N \to \infty
\]

then

\[
\limsup_{r \to \infty} \log m(r, f)/r^\rho = \infty,
\]

where \( f \) is the canonical product (8); and the values of \( r \) for which \( \log m(r, f) > \Delta r^\rho \) where \( \Delta \) is any arbitrarily large constant form a set of upper density greater than \( 1 - 1/\lambda - \epsilon \).

**Theorem 3.** Hypothesis: Let \( \rho > 0 \) be nonintegral and (1), (2), (3), (4), and (5) hold.\(^9\)

Conclusion:

(10) Any integral function of order \( \rho \) with exactly these zeros will be of the form

\[
F(z) = e^{g(z)} P(z) \prod_{n=1}^{\rho} \prod_{\nu=1}^{\mu_n} \left( 1 - \frac{z^\nu}{a_n^{\nu}} \right),
\]

where \( g(z) \) is a polynomial of degree not exceeding \( \rho \), \( P(z) \) a polynomial;\(^4\) and the values of \( r \) for which

\[
\log m(r, F) > (1 - \epsilon) \log M(r, F)
\]

holds will form a set of upper density greater than \( 1 - 1/\lambda - \epsilon \).

**Theorem 4.** If \( \rho > 0 \) and (1), (2), (3), (4), (5), and (9) hold\(^5\) then conclusion (10) holds.

**Theorem 5.** If (1), (2), (3), (4), (5), and (6) hold and if \( m_\epsilon(r) \) and \( M_\epsilon(r) \) denote the lower and upper bounds of \( |f(z)| \), where \( f(z) \) is the canonical product (8), of order \( \rho \) \((0 \leq \rho < \infty)\) in the annulus \( r \leq |z| \leq r + r^\sigma \) \((\sigma < 1 - \rho)\) then the values of \( r \) for which\(^6\)

\[\]

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\(^8\) For instance \( R_n = 2^{n^2}, \, p_n = n, \, \mu_n = 2^{(n-1)} \). Here \( \rho = 1/2 \).

\(^9\) \( P(z) \) is a polynomial having zeros at points \( a_n \exp(2\pi i k/\mu_n), \, r = 1, 2, \cdots, p_n; \)
\( \nu = 0, 1, 2, \cdots, \mu_n - 1 \) and \( n = 1, 2, \cdots, \mu_1 - 1 \) only.

\(^6\) See footnotes 2 and 3.

where \( C_1 = C_1(\lambda, \epsilon) > 0 \), holds for a set of upper density greater than \( 1 - 1/\lambda - \epsilon \).

**Proof of Theorem 1.** Let \( |z| = R = \lambda \gamma R_k \) \((0 < \gamma < 1)\), where \( k \) is so large that

\[
\lambda \gamma R_k < R_{k+1} - R_{k+1}^\alpha,
\]

\[ f(z) = P_1P_2, \]

where

\[
P_1 = \prod_{n=1}^{k} \prod_{s=1}^{p_n} \left( 1 - \frac{z^{\mu_n}}{a_{n,s}} \right),
\]

\[
P_2 = \prod_{n=k+1}^{\infty} \prod_{s=1}^{p_n} \left( 1 - \frac{z^{\mu_n}}{a_{n,s}} \right),
\]

\[
|P_1| \leq \prod_{n=1}^{k} \prod_{s=1}^{p_n} \left( 1 + \frac{R_{n+1}}{a_{n,s}^{\mu_n}} \right) \left( \prod_{n=1}^{k} \prod_{s=1}^{p_n} \left[ 1 + \frac{|a_{n,s}|^{\mu_n}}{R_{n+1}} \right] \right) = P_{11}P_{12},
\]

say. Now \(|a_{n,s}| < R_n\),

\[
|P_{12}| \leq \prod_{n=1}^{k} \left( 1 + \frac{R_{n+1}}{R_n} \right)^{p_n},
\]

and \( R_n/R \leq 1/\lambda \gamma < 1 \) for \( n = 1, 2, \ldots, k \), and \( \sum p_n/\lambda \gamma^{\mu_n} \) is convergent. Hence

\[
|P_{12}| \leq C_2,
\]

\[
|P_2| \leq \prod_{n=k+1}^{\infty} \prod_{s=1}^{p_n} \left( 1 + \frac{R_{n+1}}{a_{n,s}^{\mu_n}} \right),
\]

where \(|a_{n,s}| \geq a_{k+1,s} \geq R_{k+1} - R_{k+1}^\alpha\),

\[
\frac{R}{|a_{n,s}|} \leq \frac{R}{R_{k+1} - R_{k+1}^\alpha} \sim \frac{\lambda \gamma R_k}{R_{k+1}} \leq \frac{1}{\lambda^{1-\gamma}},
\]

and \( \sum p_n/\lambda^{(1-\gamma)\mu_n} \) is convergent. Hence

\[
|P_2| \leq C_3
\]

\footnote{\( C, C_1, C_2, \ldots \) denote finite positive (nonzero) constants.}
and so

\[ M(R) \leq C_2C_3 \prod_{n=1}^{k} \prod_{s=1}^{p_n} \frac{R^{\mu_n}}{|a_{ns}|^{\mu_n}}. \]

Further

\[
|P_1| = \prod_{n=1}^{k} \prod_{s=1}^{p_n} \left| 1 - \frac{z^{\mu_n}}{|a_{ns}|^{\mu_n}} \right|
\]

\[
\prod_{n=1}^{k} \prod_{s=1}^{p_n} \left\{ \frac{R^{\mu_n}}{|a_{ns}|^{\mu_n}} - 1 \right\}
\]

\[
\prod_{n=1}^{k} \prod_{s=1}^{p_n} \left( \frac{R^{\mu_n}}{|a_{ns}|^{\mu_n}} \right) \left( \prod_{n=1}^{k} \prod_{s=1}^{p_n} \left\{ 1 - \frac{|a_{ns}|^{\mu_n}}{R^{\mu_n}} \right\} \right)
\]

\[ = P_{11}P_{14} \]

say. Since \( \sum z / \lambda^{\gamma n} \) is convergent and

\[
|P_2| \geq \prod_{n=k+1}^{\infty} \prod_{s=1}^{p_n} \left\{ 1 - \frac{R^{\mu_n}}{|a_{ns}|^{\mu_n}} \right\} \geq C_5,
\]

(12) \[ m(R) \geq C_4C_5 \prod_{n=1}^{k} \prod_{s=1}^{p_n} \frac{R^{\mu_n}}{|a_{ns}|^{\mu_n}} \ldots, \]

which gives that \( m(R) \geq C_6M(R) \) where \( C_6 = C_6(\lambda, \gamma) \). Now given \( \varepsilon > 0 \) let \( \varepsilon_1 = \varepsilon \lambda^2 / (\lambda + 1 + \varepsilon \lambda) \). Writing \( \lambda^\gamma = \theta \) and \( R = \theta R_k \), where \( 1 + \varepsilon_1 \leq \theta \leq \lambda - \varepsilon_1 \) and \( k \geq K \), \( K \) being so large that \( R_k (\lambda - \varepsilon_1) < R_{k+1} - R_{k+1}^2 \), we get \( m(R) \geq C(\lambda, \varepsilon) m(R) \). This inequality holds good over a set of upper density greater than

\[
\frac{(\lambda - \varepsilon_1) - (1 + \varepsilon_1)}{\lambda - \varepsilon_1} = 1 - \frac{1}{\lambda} - \varepsilon.
\]

**PROOF OF THEOREM 2.** We know from (12) that \( m(R, f) \geq C_4C_6 X \), where

\[
X = \prod_{n=1}^{k} \prod_{s=1}^{p_n} \frac{R^{\mu_n}}{|a_{ns}|^{\mu_n}} \geq \lambda (\gamma \Sigma p_n r_n)
\]

\[
\log m(R, f) \geq \log (C_4C_6) + \log X \geq \log (C_4C_6) + \gamma \log \lambda \left( \sum_{n=1}^{b} \mu_n p_n \right)
\]

\[ > \Delta R^p \quad \text{for all large } R. \]

Hence \( \limsup_{R \to \infty} \log m(r, f) / r^p = \infty \). Further, the values of \( r \) for which \( \log m(r, f) > \Delta r^p \) form a set of upper density greater than \( 1 - 1 / \lambda - \varepsilon \).
THEOREM 3. Given \( \epsilon > 0 \), let \( \epsilon_2 = \epsilon / (2 - \epsilon) \). Since
\[
\sum \mu_n p_n / (R_n - R_n^{\epsilon_2})
\]
is divergent we have
\[
\mu_n p_n \geq R_n^{\epsilon_2}
\]
or \( n = k_1, k_2 \).

Let \( |z| = R = \lambda \gamma R_k \) (0 < \gamma < 1 and \( 1 + \epsilon_1 \leq \lambda \gamma \leq \lambda - \epsilon_1 \)), where \( k \) takes the values \( k_1, k_2, \ldots \). If \( X = \prod_{n=1}^{k} \prod_{n=1}^{\mu_n} R^R/|a_n|^{\mu_n} \) then
\[
X \geq \exp \{ \gamma \log \lambda \sum_{n=1}^{k} \mu_n p_n \}
\]
and so log \( X \geq C_0 \sum_{n=1}^{k} \mu_n p_n \geq C_0 R_k^{\epsilon_2 - \epsilon} = C_{\epsilon} R^{\epsilon - \epsilon}. \)
Choosing \( k \) and hence \( R \) sufficiently large we have, as in Theorem 1,
\[
\begin{align*}
m(R, F) &> C_0 \exp \{ \log X - C_{\epsilon} R^{\epsilon} \} , \\
\log m(R, F) &> \log C_0 + \log X - C_{\epsilon} R^{\epsilon} \\
&> (1 - \epsilon_2) \log X.
\end{align*}
\]
Similarly log \( M(R, F) < (1 + \epsilon_2) \log X \) which gives
\[
\frac{\log m(R, F)}{\log M(R, F)} > \frac{1 - \epsilon_2}{1 + \epsilon_2} = 1 - \epsilon.
\]
As in Theorem 1, this result holds for values of \( R \) forming a set of upper density greater than \( 1 - 1/\lambda - \epsilon \).

THEOREM 4 can be similarly proved.

THEOREM 5. We know that for \( |z| = R = \lambda \gamma R_k \) (0 < \gamma < 1, \( 1 + \epsilon_1 \leq \lambda \gamma \leq \lambda - \epsilon_1 \))
\[
m(R, f) \geq C_4 C_5 \prod_{n=1}^{k} \prod_{n=1}^{\mu_n} \frac{R^R}{|a_n|^{\mu_n}} .
\]
We can choose \( k \) so large that \( R' = R + R^\alpha \lambda^{\gamma + \epsilon_2} R_k \), where \( \gamma + \epsilon_2 < 1 \),
\[R' < R_{k+1} - R_{k+1}^{\alpha}.
\]
Now
\[
M(R', f) < C_{10} \prod_{n=1}^{k} \prod_{n=1}^{\mu_n} \frac{R^R}{|a_n|^{\mu_n}}
\]
and therefore
\[
\frac{m(R, f)}{M(R', f)} > \frac{C_4 C_5 \sum_{n=1}^{k} \mu_n p_n}{C_{10} R^{\gamma + \epsilon_2}}.
\]
Now \( Y = (R'/R)^{-\sum_{n=1}^{k} \mu_n p_n} = (1 + R^{\epsilon - 1})^{-\sum_{n=1}^{k} \mu_n p_n} \). Further \( \sum_{n=1}^{k} \mu_n p_n < (C_{11} \log R) R^{\epsilon + \epsilon_2} < R^{\epsilon + \epsilon_2} \) for all large \( R \). Hence \( Y > \exp \{-R^{\epsilon + \epsilon_2}\} \).
log \((1 + R^{\sigma-1})\) and \(R^{\sigma+\epsilon} \log(1 + R^{\sigma-1}) \sim R^{\sigma+\epsilon+\sigma-1} \to 0\) as \(R \to \infty\), since \(\sigma < 1 - \rho\) and \(\epsilon_\tau\) can be chosen so small that \(\sigma < 1 - \rho - \epsilon_\tau\). Hence \(Y > 1/2\) for all large \(R\) and so

\[
\frac{m(R, f)}{M(R', f)} > \frac{C_4 C_5}{2C_{10}}.
\]

Further

\[
\frac{m(R', f)}{M(R', f)} > C_{11}.
\]

Hence

\[
\frac{m_\epsilon(R)}{M_\epsilon(R)} = \min \left\{ \frac{m(R)}{M(R')}, \frac{m(R')}{M(R')} \right\} \leq \min \left\{ \frac{C_4 C_5}{2C_{10}}, C_{11} \right\} \leq C_1.
\]

The values of \(R\) for which this result holds form a set of upper density greater than \(1 - 1/\lambda - \epsilon\).

*Added in proof.* The positive numbers \(\epsilon\) and \(\epsilon_4\) are chosen so small that

\[
1/\lambda + \epsilon < 1; \quad [\rho] + \epsilon_4 < \rho.
\]

In the proof of Theorem 1 we showed that

\[
M(R) \leq C_2 C_3 R_1; \quad m(R) \geq C_4 C_6 P_{11},
\]

both relations holding for all \(R\) such that

\[
(1 + \epsilon_1)R_k \leq R \leq (\lambda - \epsilon_1)R_k \quad (k > K).
\]

Here

\[
C_3 = \prod_{n=1}^{\infty} \left\{ 1 + \left( \frac{1}{1 + \epsilon_1} \right)^{\mu_n} \right\}^{\nu_n}, \quad C_4 = \prod_{n=1}^{\infty} \left\{ 1 - \left( \frac{1}{1 + \epsilon_1} \right)^{\mu_n} \right\}^{\nu_n},
\]

\[
C_5 = \prod_{n=1}^{\infty} \left\{ 1 + \left( \frac{\epsilon_1}{2\lambda} \right)^{\mu_n} \right\}^{\nu_n}, \quad C_6 = \prod_{n=1}^{\infty} \left\{ 1 - \left( \frac{\epsilon_1}{2\lambda} \right)^{\mu_n} \right\}^{\nu_n}.
\]

If \(C = C_4 C_6 / C_2 C_5\) we have

\[
m(R) \geq C M(R),
\]

the inequality holding over a set of upper density greater than \(1 - 1/\lambda - \epsilon\). If we further suppose that \(\lambda = R_n / R_{n-1}\) \((n = 2, 3, \cdots)\), then this inequality holds good over a set of upper density greater than \(1 - \lambda \epsilon (1 + \epsilon) / (\lambda - 1)\).

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