of Lemma 3 to obtain suitable $\theta$'s for groups of the form $Z_1 \times Z_2 \times Z_3$ where $Z_i$ are cyclic of order $2^{n_i}$. However, it should be noted that if $G \cong G_1 \times G_2$, a one-to-one mapping $\theta$ of $G$ upon $G$ may be defined by

$$\theta[(x, y)] = [\theta_1(x), \theta_2(y)]$$

where $\theta_1$ and $\theta_2$ are one-to-one mappings of $G_1$ upon $G_1$ and $G_2$ upon $G_2$ respectively. Moreover $\theta$ satisfies the relationship $O(\eta) \supseteq O(\eta_1) \cdot O(\eta_2)$. Thus if $O(\eta_1) = n(G_1)$, $O(\eta_2) = n(G_2)$ we would have $O(\eta) = n(G_1 \times G_2)$ and $\theta$ is represented explicitly in terms of $\theta_1$ and $\theta_2$.

UNIVERSITY OF WISCONSIN

ON RINGS WHOSE ASSOCIATED LIE RINGS ARE NILPOTENT

S. A. JENNINGS

1. Introduction. With any ring $R$ we may associate a Lie ring $(R)$, by combining the elements of $R$ under addition and commutation, where the commutator $x \circ y$ of two elements $x, y \in R$ is defined by

$$x \circ y = xy - yx.$$ 

We call $(R)$ the Lie ring associated with $R$, and denote it by $\mathcal{R}$. The question of how far the properties of $\mathcal{R}$ determine those of $R$ is of considerable interest, and has been studied extensively for the case when $R$ is an algebra, but little is known of the situation in general. In an earlier paper the author investigated the effect of the nilpotency of $\mathcal{R}$ upon the structure of $R$ if $R$ contains a nilpotent ideal $N$ such that $R/N$ is commutative.\(^1\) In the present note we prove that, for an arbitrary ring $R$, the nilpotency of $\mathcal{R}$ implies that the commutators of $R$ of the form $x \circ y$ generate a nil-ideal, while the commutators of $R$ of the form $(x \circ y) \circ z$ generate a nilpotent ideal (cf. §3). If $R$ is finitely generated, and $\mathcal{R}$ is nilpotent then the ideal generated by the commutators $x \circ y$ is also nilpotent (cf. §4).

2. A lemma on $L$-nilpotent rings. We recall that the Lie ring $\mathcal{R}$ is said to be nilpotent of class $\gamma$ if we have

where $\mathfrak{R}_k = [\mathfrak{R}_{k-1}, \mathfrak{R}]$ is the Lie ideal of $\mathfrak{R}$ generated by all elements of the form $x \circ y$ with $x \in \mathfrak{R}$ and $y \in \mathfrak{R}_{k-1}$. If $R$ is a ring whose associated Lie ring is nilpotent of class $\gamma$ then we shall say that $R$ is $L$-nilpotent of class $\gamma$. It is well known that the lower central chain (1) has the property $[\mathfrak{R}_\lambda, \mathfrak{R}_\mu] \subseteq \mathfrak{R}_{\lambda+\mu}$ and hence in particular

$$[\mathfrak{R}_\lambda, \mathfrak{R}_\lambda] = 0 \quad \text{if } 2\lambda > \gamma.$$

We prove the following lemma.

**Lemma 1.** Let $R$ be an $L$-nilpotent ring of class $\gamma$. If $c \in \mathfrak{R}_{\gamma-1}$ and if $x, y$ are arbitrary elements of $R$ then

$$(c \circ x)(c \circ y) = 0,$$

and in particular

$$(c \circ x)^2 = 0.$$

If $c_1, c_2 \in \mathfrak{R}_{\gamma-1}$ and $c_1 \circ c_2 = 0$ then for arbitrary $x, y \in R$

$$(c_1 \circ x)(c_2 \circ y) = 0.$$

**Proof.** Consider the identity

$$(a \circ b \circ y \circ x) = (a \circ b \circ x) y + (a \circ b)(y \circ x) + b(a \circ y \circ x) + (b \circ x)(a \circ y).$$

Setting $a = b = c$ we have, since $[\mathfrak{R}_{\gamma-1}, \mathfrak{R}, \mathfrak{R}] = 0$,

$$0 = (c \circ x)(c \circ y)$$

and, if $x = y$,

$$0 = (c \circ x)^2,$$

while if $a = c_2$, $b = c_1$ and $c_1 \circ c_2 = 0$

$$0 = (c_1 \circ x)(c_2 \circ y),$$

which proves the lemma.

3. **Ideals generated by the lower central chain of $\mathfrak{R}$.** In what follows, $R$ will be an $L$-nilpotent ring, and we denote the lower central chain of $\mathfrak{R}$ as in (1). Let $\mathfrak{R}_k$, $k = 1, 2, \cdots, \gamma$, be the subring of $R$ generated by the elements of $\mathfrak{R}_k$, and let $\overline{\mathfrak{R}}_k$ be the ideal of $R$ generated by $\mathfrak{R}_k$. It is known\(^8\) that every element of $\overline{\mathfrak{R}}_k$ may be written in the form $u_k + v_k$, where $u_k \in \mathfrak{R}_k$ and $v_k \in RR_k$, and since $R_\gamma$ is in the centre of $R$, $\overline{\mathfrak{R}}_\gamma$ is a nilpotent or nil-ring whenever $R_\gamma$ is.

\(^8\) Ibid. Lemma 5.3.
Let $R^* = R/\overline{R}_\gamma$; then the natural homomorphism of $R$ upon $R^*$ induces a homomorphism of $\mathfrak{R}$ upon $\mathfrak{R}^*$, where $\mathfrak{R}^*$ is the Lie ring associated with $R^*$, such that $\overline{R}_k \rightarrow \overline{R}_{k^*}$. Hence in particular $\mathfrak{R}_{\gamma}^* = 0$ and $R^*$ is an $L$-nilpotent ring of class not greater than $\gamma - 1$.

Our principal theorem is the following:

**Theorem 1.** If $R$ is an $L$-nilpotent ring, then the commutators of $R$ generate a nil-ideal of $R$, that is, $\overline{R}_2$ is a nil-ideal. The elements of $R$ of the form $(x \circ y) \circ z$ generate a nilpotent ideal of $R$, that is, $\overline{R}_3$ is nilpotent.

**Proof.** Consider first $\overline{R}_\gamma$: every element of $\overline{R}_\gamma$ can be written as a finite sum of finite products of elements of $\mathfrak{R}_\gamma$ and since $\mathfrak{R}_\gamma = [\mathfrak{R}_{\gamma - 1}, \mathfrak{R}]$, every element of $\mathfrak{R}_\gamma$ can be written as a finite sum of elements of the form $c \circ x$, where $c \in \mathfrak{R}_{\gamma - 1}$ and $x \in R$. Hence every element of $\overline{R}_\gamma$ is a sum of products of elements of the form $c \circ x$. Now by Lemma 1 the square of every element of the form $c \circ x$ is zero, and these elements are all in the centre of $R$. Hence if

$$ y = p_1 + p_2 + \cdots + p_n $$

is an element of $\overline{R}_\gamma$, where the $p_k$ are products of elements of the form $c \circ x$, we have $p_k^2 = 0$ and therefore, since these products $p_k$ are all in the centre of $R$, we have $y^{n+1} = 0$, which proves that $\overline{R}_\gamma$, and hence $\overline{R}_\gamma$, is a nil-ring. Now if $\gamma > 2$ we have, from (2)

$$ [\mathfrak{R}_{\gamma - 1}, \mathfrak{R}_{\gamma - 1}] = 0 $$

and hence $c_1 \circ c_2 = 0$ for all $c_1, c_2 \in \mathfrak{R}_{\gamma - 1}$. From Lemma 1 it follows that $p_1p_2 = 0$ in the representation of $y$ above, and hence

$$ \overline{R}_\gamma^3 = 0, \quad \gamma > 2. $$

The proof of the theorem now proceeds easily by induction upon $\gamma$, since by the above it is true when $\gamma = 2$, that is whenever $\overline{R}_2 = \overline{R}_\gamma$, $\overline{R}_3 = 0$. We suppose, therefore, that the theorem holds for rings of class less than $\gamma$, and hence in particular for $R^* = R/\overline{R}_\gamma$. Then if $c \in \overline{R}_2$ and $c \rightarrow c^*$ in the homomorphism of $R$ upon $R^*$ we have

$$ c^{*\sigma'} = 0, \quad \sigma' \text{ some integer}, $$

by our induction, and hence

$$ c^* \in \overline{R}_\gamma \quad \text{for all } c \in \overline{R}_2. $$

Since $\overline{R}_\gamma^3 = 0$ whenever $\gamma > 2$ we have

$$ c^\sigma = 0, \quad \text{where } \sigma = 2\sigma', $$

by our induction.
and it follows that $\mathcal{R}_s$ is a nil-ring. Further, since $\mathcal{R}_s^*$ is nilpotent by our induction,

$$\mathcal{R}_s^{*r'} = 0$$

for some integer $r'$ and hence

$$\mathcal{R}_s^{r'} \subseteq \mathcal{R},$$

and therefore

$$\mathcal{R}_s^\tau = 0,$$

where $\tau = 2r'$, which proves that $\mathcal{R}_s$ is nilpotent, as required.

4. Finitely generated $L$-nilpotent rings. If $\mathcal{R}$ satisfies the maximal or minimal condition for one-sided ideals, so does $\mathcal{R}_s$ and hence $\mathcal{R}_s$ must be nilpotent.\(^4\) We prove the following stronger result:

**Theorem 2.** If $\mathcal{R}$ is a finitely generated $L$-nilpotent ring, then the commutators of $\mathcal{R}$ generate a nilpotent ideal, that is, $\mathcal{R}_s$ is nilpotent.

**Proof.** If $\mathcal{R}$ is finitely generated, say by $x_1, x_2, \cdots, x_d$, then every element $x$ of $\mathcal{R}$ can be written in the form $x = p_1 + p_2 + \cdots + p_n$ where the $p_k$ are products of the $x_1, \cdots, x_d$ in some order. It is clearly sufficient to consider the case $\gamma = 2$, since if we show in general that $\mathcal{R}_s/\mathcal{R}_s^r$ is nilpotent, it will follow from Theorem 1 that $\mathcal{R}_s$ has this property. Because of the identity

$$ (ab) \circ c = a(b \circ c) + (a \circ c)b $$

every element of $\mathcal{R}_s$ can be written as a sum of products of the form

$$ \pi_r = a(x_{i_1} \circ x_{i_2})(x_{i_3} \circ x_{i_4}) \cdots (x_{i_r} \circ x_{i_r}), \quad a \in \mathcal{R}. $$

Now there are at most $d(d-1)/2$ nonzero commutators of the type $x_i \circ x_j$, and since by Lemma 1 we have

$$ (x_i \circ x_j)(x_i \circ x_k) = 0 $$

it follows that if the number of factors in any product $\pi_r$ is greater than $d(d-1)/2$ this product vanishes. Hence

$$ \mathcal{R}_s^\tau = 0, \quad \tau = d(d - 1)/2 + 1 $$

and the theorem is established.

---

In connection with Theorem 2 it would be of interest to know if there exist $L$-nilpotent rings for which $R_3$ is not nilpotent. It would be enough to exhibit a ring $R$ for which

$$(x \circ y) \circ z = 0$$

for all $x, y, z \in R$ and such that the subring generated by elements of the form $(x \circ y)$ is not nilpotent. The author has been unable to construct such a ring but it seems fairly safe to conjecture that such a one exists, and indeed with a countable generating set.

Since $R/R_3$ is commutative and $R_3$ is nilpotent we have at once from an earlier result of the author:

**Theorem 3.** A finitely generated $L$-nilpotent ring is of finite class.

Finally, it is clear that we have the following criterion for the nilpotency of a finitely generated nil-ring:

**Theorem 4.** A finitely generated nil-ring is nilpotent if and only if its associated Lie ring is nilpotent.

This last theorem may be compared with Kaplansky's result on finitely generated nil-algebras, which states that, provided the ground field has enough elements, such an algebra is nilpotent if and only if there exists a fixed integer $p$ such that $x^p = 0$ for all elements $x$ in the algebra. Our theorem shows that this condition may be replaced by the requirement that all commutators of a fixed weight vanish.

**The University of British Columbia**

---