SOME REMARKS AND CORRECTIONS TO
ONE OF MY PAPERS

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Professor Hartmann pointed out two inaccuracies in my paper Some remarks about additive and multiplicative functions (Bull. Amer. Math. Soc. vol. 52 (1946) pp. 527–537) (see Mathematical Reviews vol. 7 (1946) p. 577).

His first objection is that my proof of Theorem 12 (see p. 535) assumes that \( f(p^a) \geq 0 \). The only place the error occurs is in the fifth formula line of p. 536. But the error is quite easy to correct, only a \( O(1) \) term is missing. The correct version of the formula is

\[
\frac{\sum_{m=1}^{n} g_k(m)}{n \sum_d \frac{h_k(d)}{d}} + O(1) < n \prod_p \left(1 + \frac{h_k(p)}{p}\right) + O(1).
\]

Otherwise the proof is unchanged.

His second objection is against Theorem 13 (pp. 536–537) and is more serious.

Theorem 13 was stated as follows: Let \( g(n) \geq 0 \) be multiplicative. Then the necessary and sufficient condition for the existence of the distribution function is that

\[
\sum_p \frac{(g(p) - 1)'}{p} < \infty, \quad \sum_p \frac{(g(p) - 1)'^2}{p} < \infty
\]

where \((g(p) - 1)' = g(p) - 1\) if \(|g(p) - 1| \leq 1\) and 1 otherwise.

I try to prove this by putting \( \log g(n) = f(n) \) and state that \( g(n) \) has a distribution function if and only if \( f(n) \) has a distribution function.

In his review Hartmann points out that first of all this implies \( g(n) > 0 \) (instead of \( g(n) \geq 0 \)), and in a letter he points out that my statement is incorrect if \( g(n) \) has a distribution function but \( \lim_{x \to 0} G(x) > 0 \) (\( G(x) \) being the distribution function of \( g(x) \)). (I seem to remember that in my mind I was somehow unwilling to admit these \( G(x) \) as distribution functions, but neglected to state this.)

In fact it is easy to see that this case can occur. Put \( g(p^\alpha) = 1/2 \) for all \( p \) and \( \alpha \). Then \( G(x) = 1 \) for all \( x \geq 0 \), but clearly \( f(n) \) has no distribution function, and the series (1) do not converge. Thus Theorem 13 is incorrect as it stands. The correct version may be stated as follows:

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THEOREM 13'. Let \( g(n) \geq 0 \) be multiplicative. Assume that the series (1) converge. Then \( g(n) \) has a distribution function. The converse is also true unless \( G(x) = 1 \) for all \( x \geq 0 \).

First of all we remark that if

\[
\sum_{p \in \mathbb{P}, g(p) = 0} \frac{1}{p} = \infty
\]

we have \( G(x) = 1 \) for all \( x \geq 0 \) (since almost all integers are divisible by a \( p \) with \( g(p) = 0 \)). Thus this case can be neglected, and we can assume that the primes with \( g(p) = 0 \) can be neglected, since they do not influence the convergence of the series (1) or the existence of the distribution function.\(^1\)

The first part of Theorem 13 follows as on p. 537 of my paper. Next we investigate the converse. If we assume that \( \lim_{x \to +0} G(x) = 0 \) the convergence of (1) follows as on p. 537, since in this case it really is true that \( g(n) \) has a distribution function if and only if \( f(n) \) has a distribution function.

Assume now

\[(2) \quad \lim_{x \to +0} G(x) = c > 0.\]

We shall show \( c = 1 \). Suppose that \( c < 1 \), we shall show that this leads to a contradiction.

Denote by \( F(x) \) the density of integers with \( f(n) < x \) (where \( f(n) = \log g(n) \)). Clearly \( F(x) \) exists and satisfies (\( G(x) \) is a distribution function)

\[
\lim_{x \to +\infty} F(x) = c > 0, \quad \lim_{x \to -\infty} F(x) = 1 \quad (c < 1). \tag{3}
\]

From now on we make constant use of my joint paper with Wintner\(^1\) (referred to as E.W.). It follows from (3) that there exist real numbers \( a \) and \( b \) such that

\[
\infty < a < b < \infty \quad \text{and} \quad F(b) - F(a) > 0. \tag{4}
\]

From (4) and E.W. §9, p. 717 it follows that \( |f(p)| < A \) (except for a sequence of primes \( q \) with \( \sum 1/q < \infty \), which can be neglected).

Next we deduce (E.W. §3, pp. 714–715) that

\[
\sum_p \frac{(f(p))^2}{p} < \infty. \tag{5}
\]

Further it follows that (E.W. §4, p. 714)

\[
\left\lvert \sum_{p \leq x} \frac{f(p)}{p} \right\rvert < B \quad \text{(B independent of } x)\text{).}
\]

In §6, p. 716 it is shown that from \(|f(p)| < A\), (4) and (5) it follows that

\[
\sum_{m=1}^{n} (f(m))^2 < Cn.
\]

But clearly (7) contradicts (3) (since (3) implies that the density of integers with \(f(m) > D\) is not less than \(c\) for every \(D\)), which completes the proof of Theorem 13'.

The following question can be raised: Let \(f(n)\) be additive and assume that for some \(a < b\) the density of the integers satisfying \(a \leq f(n) \leq b\) exists and is different from 0. Does it then follow that \(f(n)\) has a distribution function?

By the same methods as just used we can show that

\[
|f(p)| < c, \quad \sum_{p} \frac{(f(p))^2}{p} < \infty, \quad \sum_{p} \frac{f(p)'}{p} < \infty.
\]

But at present I cannot decide whether the distribution function has to exist.

Professor Hartmann also pointed out the following misprints in my previous paper:

1. The first sentence of Theorem 12 should read “Let \(f(p^a) \leq C\).”
2. The inequality symbol in the two formula lines at the bottom of p. 535 should be “\(\leq\)” instead of “\(>\).”
3. On p. 537, in the line following the third formula line “\((\log g(p))^2 > \cdots\)” should be “\((\log g(p))^2 > \cdots\).”
4. On p. 537, the fifth formula line should be “\(\sum (1/p) \cdots\)” instead of “\(\sum(1/2) \cdots\).”
5. In the next to the last line of the paper, p. 537, “\(\cdots f(n)\)” should be “\(\cdots g(n)\).”
6. The first formula on p. 529 should read “\(\cdots \exp \exp (d\phi(n))\)” instead of “\(\cdots \exp \exp (\phi(n))\).”

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