NOTE ON POWER SERIES

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1. The problem. The following question was raised by Bochner. Let $\sum \alpha_{ik} \xi^k \eta^k$ be a power series with complex coefficients, such that substitution of convergent power series $\sum \alpha_i \xi^i$ and $\sum \beta_i \xi^i$ for $\xi$ and $\eta$ produces always a convergent power series in $\xi$. Is the double series $\sum \alpha_{ik} \xi^k \eta^k$ convergent?

The answer is yes; we present a proof which presupposes from function theory only the Cauchy estimate for the coefficients of polynomials in a complex variable:

\[(C) \quad |\gamma_{ik}^k| \leq (|\xi| = |\xi_0|) \sup |\sum \gamma_{ik}^k|.
\]

We note that this estimate is also valid in certain types of fields with non-Archimedian valuations, namely, those for which the values are dense and the index is infinite; this was shown by Schoebe in \[1].\[1]

2. Homogeneous polynomials. We denote a vector $(\xi, \eta)$ by $x$ and introduce as the norm $\|x\|$ of $x$ the maximum of $|\xi|$ and $|\eta|$. A complex Banach space results which, as a complete metric space, is of the second category with respect to itself. We then consider homogeneous polynomials $P(x) = \sum_{i+k=n} \alpha_{ik} \xi^i \eta^k$; it is clear that $P(x+\xi_0) = \xi^* P(x)$, that $P(x+\xi x_0)$ is a polynomial in $\xi$, and that $P$ is a continuous function of $x$.

The following three lemmata are immediate consequences of the estimate (C).

(2.1) LEMMA. If $|P(x)| \leq M \text{ for } \|x\| \leq \xi$, then $|\alpha_{ik} \xi^i \eta^k| \leq M \text{ for } |\xi|, |\eta| \leq \xi$.

(2.2) LEMMA. $|P(x)| \leq (|\xi| = 1) \sup |P(x+\xi x_0)|$.

This special case of the principle of the maximum is a special case of (C), applied to the constant term of $P(x+\xi x_0)$, considered as a polynomial in $\xi$. It is used in the proof of (2.3).

(2.3) LEMMA. If $|P(x)| \leq M \text{ for } \|x-x_0\| \leq \xi$, then $|P(x)| \leq M \text{ for } \|x\| \leq \xi$.

PROOF (compare $[2, \text{ p. 590}]$): $|P(x)| \leq (|\xi| = 1) \sup |P(\xi x_0 + x)| = (|\xi| = 1) \sup |P(x_0 + \xi^{-1} x)| \leq (\|x_1 - x_0\| \leq \|x\|) \sup |P(x)|$.

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\[1\] Numbers in brackets refer to the references cited at the end of the paper.

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3. Proof of the theorem. We may now dispose of the problem by proving a slightly stronger result.

\[(3.1)\text{ Theorem. If the substitution } \xi = \alpha \xi, \eta = \beta \eta \text{ produces from } \sum \alpha_{ik} \xi^i \eta^k \text{ always a power series with a nonvanishing radius of convergence then the series } \sum |\alpha_{ik} \xi^i \eta^k| \text{ converges for sufficiently small } |\xi| \text{ and } |\eta|.

Proof. The result of the substitution is, with \((\alpha, \beta) = a\),

\[
\sum_{n=0}^{\infty} \left( \sum_{i+k=n} \alpha_{ik} \alpha^i \beta^k \right) \xi^n = \sum_{n=0}^{\infty} P_n(a) \xi^n.
\]

Now let \(\delta\) be a complex number for which \(0 < |\delta| < 1\); there will exist, for every vector \(a\), an integer \(m\) such that \(\sum P_n(a)(\delta^m)^n = \sum P_n(a\delta^m)\) converges. We say that the set \(D\) of the vectors \(x\) for which \(\sum P_n(x)\) converges is of the second category. For every vector is in one of the sets \(\delta^{-m}D\); if \(D\) were of the first category, the sets \(\delta^{-m}D\) and therefore the whole space would be of the same character.

By virtue of the continuity of the functions \(P_n\) there will exist (compare [3, p. 19]) a sphere \(\|x - x_0\| \leq \rho, \rho > 0\), and an \(M\) such that \(\|P_n(x)\| \leq M\) holds in it for all \(n\). By Lemma (2.3), the same inequality is valid for \(\|x\| \leq \rho\); therefore, \(\|P_n(x)\| \leq M/2^n\) will be true for \(\|x\| \leq \rho/2\). By Lemma (2.1) we have, for \(\|\xi\|, |\eta| \leq \rho/2\), the inequalities

\[
|\alpha_{ik} \xi^i \eta^k| \leq M/2^n = M/2^{i+k},
\]

which establish the absolute convergence of the double series.

4. Comments. The main point of our arrangement was the weakening of the premise; this procedure was possible essentially because we worked with the complex numbers. It would be interesting to know whether the original conjecture holds in the real case.

References


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\(^3\) The introduction of the powers of a number in the place of the integers is a concession to the non-Archimedean case and otherwise not relevant.

\(^4\) This inference can also be made in the non-Archimedean case, provided the valuation is dense.