CONGRUENCE PROPERTIES OF RAMANUJAN'S FUNCTION $\tau(n)$

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Introduction. With Ramanujan we define $\tau(n)$ by

$$\sum_{1}^{\infty} \tau(n) x^n = x \prod_{1}^{\infty} (1 - x^n)^{24} \quad (|x| < 1).$$

Write $\sigma_k(n)$ for the sum of the $k$th powers of the divisors of $n$; $\sigma(n) = \sigma_1(n)$. It is known that$^1$

$$\tau(n) \equiv n\sigma(n) \pmod{5},$$

$$\tau(n) \equiv \sigma(n) \pmod{3} \quad \text{if } (n, 3) = 1.$$ 

The object of this note is to give proofs of the much stronger results:

(A) $\tau(n) \equiv 5n^2\sigma_7(n) - 4n\sigma_9(n) \pmod{5^3}$

when $n$ is prime to 5;

(B) $\tau(n) \equiv (n^2 + k)\sigma_7(n) \pmod{3^4}$

when $n$ is prime to 3 and where $k = 0$ if $n \equiv 1(3)$, $k = 9$ if $n \equiv 2(3)$.

1. Some lemmas.

**Lemma 1.** We have

$$\sum u\sigma_3(u)\sigma_5(v) = \sum \sigma(u)\sigma(v) - P(n) \pmod{5}$$

where

$$P(n) = \sum_{u \equiv 0 \pmod{5}} \sigma(u)\sigma(v)$$

where $u + v = n$; $u, v \geq 1$ in all three sums ($\sum$).

**Proof.** We have

(1) $u\sigma_3(u)\sigma_5(v) \equiv 0 \pmod{5}$ when $u \equiv 0(5)$;

when $(u, 5) = 1$ we have

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$^1$ The first of these is proved in Hardy's Ramanujan (Cambridge, 1940); the second by Gupta in J. Indian Math. Soc. vol. 9 (1945) pp. 59–60. In what follows we refer to Ramanujan's Collected papers (Cambridge, 1927) by the letters RCP. We have also proved that $\tau(n) = \sigma_1(n) \pmod{2^8}$ if $n$ is odd; this result has been accepted for publication in J. London Math. Soc.
\[ u \sigma_3(u) = u \sum_{d \mid u} d^3 = u \sum_{d \mid u} d^{-1} = \sum_{d \mid u} \frac{u}{d} = \sigma(u), \]

so that

\[ (2) \quad u \sigma_3(u) \equiv \sigma(u) \pmod{5} \quad \text{when } (u, 5) = 1. \]

Similarly

\[ (3) \quad \sigma_3(v) \equiv \sigma(v) \pmod{5}. \]

From (1), (2), (3):

\[ \sum u \sigma_3(u) \sigma_3(v) \equiv \sum_{(u, 5) = 1} \sigma(u) \sigma(v) \equiv \sum \sigma(u) \sigma(v) - P(n) \pmod{5}. \]

**Lemma 2.** If \((n, 5) = 1\) we have

\[ \sum u \sigma_3(u) \sigma_3(v) \equiv \sum \sigma(u) \sigma(v) - 2P(n) \pmod{5} \]

where, as in Lemma 1, the conditions

\[ u + v = n, \quad u, v \geq 1 \]

are understood in every \( \sum \).

**Proof.** If \( u \) or \( v \equiv 0(5) \), \( u \sigma_3(u) \sigma_3(v) \equiv 0(5) \). From this and (1) we get since \((n, 5) = 1\),

\[ \sum u \sigma_3(u) \sigma_3(v) \equiv \sum_{(u, 5) = 1, (v, 5) = 1} \sigma(u) \sigma(v) \equiv \sum \sigma(u) \sigma(v) - 2P(n) \pmod{5}, \]

the desired result.

2. **Proof of (A).** Write, for \( x \) numerically less than unity,

\[ P = 1 - 24 \sum_{1}^{\infty} \sigma(n) x^n, \]

\[ Q = 1 + 240 \sum_{1}^{\infty} \sigma_3(n) x^n, \]

\[ R = 1 - 504 \sum_{1}^{\infty} \sigma_6(n) x^n. \]

Then (44), p. 144 of RCP, is

\[ (4) \quad Q^3 - R^3 = 1728 \sum_{1}^{\infty} \tau(n) x^n \]

and we deduce from relations 5 and 2, Table II, p. 142 of RCP, that
Comparing coefficients of $x^n$ and using Lemma 1 we have

\[ 1584 \sum_{n=1}^{\infty} n \sigma_9(n) x^n \]
\[ = 3(Q^3 - R^2) - 5R(PQ - R) \]
\[ = 5184 \sum_{n=1}^{\infty} \tau(n) x^n - 5 \left( 1 - 504 \sum_{n=1}^{\infty} \sigma_6(n) x^n \right) (PQ - R) \]
\[ = 5184 \sum_{n=1}^{\infty} \tau(n) x^n - 5 \left( 1 - 504 \sum_{n=1}^{\infty} \sigma_6(n) x^n \right) 720 \sum_{n=1}^{\infty} n \sigma_3(n) x^n. \]

Comparing coefficients of $x^n$ and using Lemma 1 we have

\[ 1584 n \sigma_9(n) = 5184 \tau(n) - 3600 n \sigma_9(n) + 5 \cdot 504 \cdot 720 \sum u \sigma_3(u) \sigma_3(v) \]
(where $u + v = n$ ($u, v \geq 1$) in the $\sum$ sum),

\[ 84 n \sigma_9(n) = 59 \tau(n) + 25 n \sigma_9(n) \]
\[ + 25 \sum \sigma(u) \sigma(v) - 25 P(n) \pmod{125}. \]

Again, relations 4, Table III, and 2, Table II, p. 142 of RCP give us

\[ 8640 \sum_{n=1}^{\infty} n^2 \sigma_7(n) x^n \]
\[ = 5(Q^3 - R^2) + 9(PQ - R)^2 \]
\[ = 8640 \sum_{n=1}^{\infty} \tau(n) x^n + 9 \cdot 720^2 \left\{ \sum_{n=1}^{\infty} n \sigma_3(n) x^n \right\}^2. \]

Comparing the coefficients of $x^n$ here we get

\[ n^2 \sigma_7(n) = \tau(n) + 135 \cdot 4 \sum u \sigma_3(u) \sigma_3(v) \]
or

\[ 15 n^2 \sigma_7(n) = 15 \tau(n) - 25 \sum \sigma(u) \sigma(v) + 50P(n) \pmod{125}. \]

From \(7'\) and Lemma 2 we get

\[ 15 n^2 \sigma_7(n) = 15 \tau(n) - 25 \sum u \sigma_3(u) \sigma_3(v) \pmod{125}. \]

Eliminating $P(n)$ from (6) and (8) we get

\[ 168 n \sigma_9(n) + 15 n^2 \sigma_7(n) = 135 \tau(n) + 50 n \sigma_9(n) \]
\[ + 25 \sum \sigma(u) \sigma(v) \pmod{125}, \]
or

\[ 8 \tau(n) = 43 n \sigma_9(n) + 15 n^2 \sigma_7(n) - 50 n \sigma_9(n) \]
\[ - 25 \sum \sigma(u) \sigma(v) \pmod{125}. \]

Again (relation 1, Table IV, p. 146 of RCP)
\[
\sum \sigma(u)\sigma(v) = \frac{5\sigma_3(n) - 5n\sigma(n)}{12} - \frac{(n - 1)\sigma(n)}{12}
\]
\[
= 2(n - 1)\sigma(n) \pmod{5}.
\]

From (9) and (10) we obtain
\[
8\tau(n) \equiv 43n\sigma_3(n) + 15n^2\tau(n)
- 50n\sigma_3(n) - 50(n - 1)\sigma(n) \pmod{5^3}.
\]

Hence, multiplying by 47,
\[
\tau(n) \equiv 21n\sigma_3(n) - 45n^2\tau(n) + 25n\sigma_3(n)
+ 25(n - 1)\sigma(n) \pmod{5^3}
\]
\[
\equiv 5n^2\tau(n) - 4n\sigma_3(n) \pmod{5^3}
\]
for
\[
25n\sigma_3(n) - 50n^2\tau(n) + 25n\sigma_3(n) + 25(n - 1)\sigma(n)
= 0 \pmod{125},
\]
since the terms inside each set of braces are a multiple of 5 provided \((n, 5) = 1\). Thus (A) is proved by (11).

3. Proof of (B). We shall need the following results:
\[
\sigma(3t + 2) \equiv 0 \pmod{3},
\]
\[
\sigma_3(3t + 2) \equiv 0 \pmod{9}
\]
where \(t\) is 0 or a positive integer. To prove (13) we observe that to every divisor \(3m + 1\) of \(3t + 2\), there corresponds another \(3m + 2\) satisfying \(3m + (3n + 2)/(3m + 1)\), and
\[
(3m + 1)^3 + (3n + 2)^3 \equiv 0 \pmod{9};
\]
while (12) is proved still more simply. We next prove the following lemma.

**Lemma 3.** If \(n \equiv 1 \pmod{3}\), we have
\[
\sum uv\sigma_3(u)\sigma_3(v) \equiv 0 \pmod{3}
\]
where (in the summation \(\sum\)) \(u + v = n\) and \(u, v \geq 1\).

**Proof.** Since \(n \equiv 1 \pmod{3}\) and \(u + v = n\), we have the 3 cases:
so that \( uv_{\sigma_3}(u)\sigma_3(v) \equiv 0(3) \) in each case on account of (13). Hence the lemma is proved.

**Lemma 4.** If \( n \equiv 2(3) \), we have

\[
\sum_{u+v=n, u, v \geq 1} uv_{\sigma_3}(u)\sigma_2(v) \equiv \frac{\sigma_7(n) - \sigma_3(n)}{120} \pmod{3}.
\]

**Proof.** If \( u+v=n, n \equiv 2(3) \), we have 3 cases:

(i) \( u \equiv 0(3), \quad v \equiv 2(3), \)

(ii) \( u \equiv 2(3), \quad v \equiv 0(3), \)

(iii) \( u \equiv 1(3), \quad v \equiv 1(3). \)

In the first two cases

\[ uv_{\sigma_3}(u)\sigma_3(v) \equiv 0 \pmod{3}; \]

while in the third case

\[ uv_{\sigma_3}(u)\sigma_3(v) \equiv \sigma_3(u)\sigma_3(v) \pmod{3}. \]

Hence we have (in the sums \( u+v=n; u, v \geq 1 \)), using (13),

\[
\sum uv_{\sigma_3}(u)\sigma_3(v) = \sum \sigma_3(u)\sigma_3(v) \pmod{3} = \sum \sigma_3(u)\sigma_3(v) \equiv \frac{\sigma_7(n) - \sigma_3(n)}{120} \pmod{3}
\]

since (relation 3, Table IV of RCP, p. 146)

\[
\sum \sigma_3(u)\sigma_3(v) = \frac{\sigma_7(n) - \sigma_3(n)}{120}
\]

where, in the \( \sum, u+v=n \) (\( u, v \geq 1 \)).

We are now ready to prove (B). Comparing the coefficients of \( x^n \) in (6') we obtain

(15) \[ 27.320n^2\sigma_7(n) = 27.320\tau(n) + 27^2 \cdot 80^2 \sum uv_{\sigma_3}(u)\sigma_3(v). \]

We, therefore, have

(16) \[ \tau(n) \equiv n^2\sigma_7(n) \pmod{3}. \]
Case 1. $n \equiv 1(3)$. In this case (15) and Lemma 3 give

\begin{equation}
\tau(n) \equiv n^2\sigma_7(n) \pmod{3^4}.
\end{equation}

Case 2. $n \equiv 2(\text{mod } 3)$. In this case (15) and Lemma 4 give

\begin{equation}
\tau(n) \equiv n^2\sigma_7(n) - \frac{27 \cdot 20}{120} \left\{ \sigma_7(n) - \sigma_3(n) \right\} \pmod{3^4}
\end{equation}

or

\begin{equation}
\tau(n) \equiv (n^2 + 36)\sigma_7(n) \pmod{3^4}
\end{equation}

\begin{equation}
\equiv (n^2 + 9)\sigma_7(n) \pmod{3^4}
\end{equation}

since, when $n \equiv 2(3)$, we have

\begin{align*}
\sigma_7(n) &= \sigma(n) = 0(3), \\
\sigma_3(n) &= 0(9),
\end{align*}

from (12) and (13).

(17) and (18) together give (B).

Mordell proved Ramanujan's conjecture

\begin{equation}
\tau(mn) = \tau(m)\tau(n) \quad \text{if } (m, n) = 1.
\end{equation}

From this result or directly we can prove that

(C) \quad \tau(n) \equiv 16n\sigma_6(n) \pmod{5^6} \quad \text{if } n \equiv 0(5),

(D) \quad \tau(n) \equiv n^2\sigma_7(n) \pmod{3^4} \quad \text{if } n \equiv 0(3).