A NOTE ON THE SCHMIDT-REMAK THEOREM

FRED KIOKEMEISTER

Let $G$ be a group with operator domain $\Omega$. We shall say that $G$ satisfies the modified maximal condition for $\Omega$-subgroups if the chain $H_1 \subset H_2 \subset \cdots \subset H \neq G$ is finite whenever $H_1$, $H_2$, $\cdots$, $H$ are $\Omega$-subgroups of $G$.

Let $A_1, A_2, \cdots$ be a countable set of groups. The direct product of $A_1, A_2, \cdots$ will be defined to be the set of elements $(a_1, a_2, \cdots)$ where $a_i$ is an element of $A_i$ for $i = 1, 2, \cdots$, and where but a finite number of the $a_i$ are not the identity elements of the groups in which they lie. A product in the group is defined by the usual componentwise composition of two elements. This group will have the symbol $A_1 \times A_2 \times \cdots$.

The following theorem is in a sense a generalization of the Schmidt-Remak theorem.

**Theorem.** Let $G$ be a group with operator domain $\Omega$, and let $\Omega$ contain the inner automorphisms of $G$. Let $G = A_1 \times A_2 \times \cdots$ where each of the $\Omega$-subgroups $A_i$ is directly indecomposable, and each satisfies the minimal condition and the modified maximal condition for $\Omega$-subgroups. Then if $G = B_1 \times B_2 \times \cdots$ is a second direct product decomposition of $G$ into indecomposable factors, the number of factors will be the same as the number of the $A_i$. Further the $A_i$ may be so rearranged that $A_i \cong B_i$, and for any $j$

$$G = B_1 \times B_2 \times \cdots \times B_j \times A_{j+1} \times A_{j+2} \times \cdots.$$ 

A proof of the theorem can be based on any standard proof of the Schmidt-Remak theorem such as that given by Jacobson\(^1\) or by Zassenhaus\(^2\) with but slight changes in the two fundamental lemmas.

We state the following lemmas for a group $G$ with operator domain $\Omega$, and we assume that for $G$ and $\Omega$:

- (1) $\Omega$ contains all inner automorphisms of $G$.
- (2) $G$ satisfies the minimal condition and the modified maximal condition for $\Omega$-subgroups.
- (3) $G$ is indecomposable into the direct product of $\Omega$-subgroups.

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LEMMA 1. Let \( \alpha \) be an \( \Omega \)-operator of \( G \). If there exists in \( G \) an element \( h \) not equal to the identity of \( G \) such that \( h^\alpha = h \), then \( \alpha \) is an automorphism of \( G \).

This lemma follows by the usual arguments. It is only necessary to note that the fixed point \( h \) is sufficient to guarantee that the union of the kernels of the operators \( \alpha, \alpha^2, \cdots \) is not \( G \), and that the modified maximal condition then yields that this union is the kernel of some \( \alpha^k \).

LEMMA 2. Let \( \alpha_1, \alpha_2, \cdots \) be addible \( \Omega \)-operators such that if \( g \) is an element of \( G \), then there exists an integer \( N(g) \) such that \( g^\alpha = e \), the identity element of \( G \), for all \( i > N(g) \). If \( \alpha = \alpha_1 + \alpha_2 + \cdots \) is an automorphism of \( G \) then, for some \( k \), \( \alpha_k \) is an automorphism of \( G \).

Let \( g \) be an element of \( G \), \( g \neq e \). Let \( \beta_1 = \alpha_1 + \alpha_2 + \cdots + \alpha_{N+1} = \alpha_N + \alpha_{N+2} + \cdots \) where \( N = N(g) \). Thus \( \alpha = \beta_1 + \beta_2 \) and \( g^{\beta_2} = e \).

We may assume that \( \alpha \) is the identity operator. Then \( g = g^\alpha = g^{\beta_1} g^{\beta_2} = g^{\beta_1} \). The group \( G \) and the operator \( \beta_1 \) satisfy the conditions of Lemma 1, and \( \beta_1 \) is an automorphism of \( G \).

Similarly let \( \gamma = \alpha_1 + \alpha_2 + \cdots + \alpha_{N-1} \). Then \( \beta_1 = \gamma + \alpha_N \). We may assume that \( \beta_1 \) is the identity operator. If \( \alpha_N \) is not an automorphism of \( G \), the kernel of \( \alpha_N \) must contain an element \( h \neq e \), since \( G \) satisfies the minimal condition. Again we may show that \( \gamma \) is an automorphism of \( G \). A repetition of this argument establishes the lemma.

By reference to Lemma 2 the cited proofs of the Schmidt-Remak theorem can be made to yield the following: To each \( B_i \) there corresponds a group \( A_{a_i} \) where \( a_i \) is a positive integral subscript such that \( \alpha_i = \alpha_j \) implies \( i = k \) and \( A_{a_i} \) is operator isomorphic with \( B_i \) for all \( i \). Further

\[
G = B_1 \times B_2 \times \cdots \times B_i \times A_{\beta_1} \times A_{\beta_2} \times \cdots
\]

where \( \beta_n \neq \alpha_i \) for any \( n \) or \( i \), and where the set of integers \( \{ \alpha_1, \alpha_2, \cdots, \alpha_i, \beta_1, \beta_2, \cdots \} \) is the set of all positive integers. Let \( A_m \) contain the element \( g \neq e \). Then for some \( M \), \( g \) is an element of the group

\[
(B_1 \times B_2 \times \cdots \times B_M) \cap (A_{\beta_1} \times A_{\beta_2} \times \cdots ) = e,
\]

\( m \neq \beta_k \) for all \( k \). Thus for some \( i \), \( 1 \leq i \leq M \), we have \( m = \alpha_i \), and the set of integers \( \{ \alpha_1, \alpha_2, \cdots \} \) includes all subscripts. There then exists a reordering of these subscripts such that \( \alpha_i = i \).

Mount Holyoke College