following lower bounds for \( x \) and \( \phi(x) \): (I) \( 10^{458} \); (II) \( 10^{588} \); (III) \( 10^{400} \).

REFERENCES


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ON THE DARBOUX TANGENTS

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1. Introduction. In a recent paper [1] Abramescu gave a metrical characterization of the cubic curve obtained by equating to zero the terms of the expansion of a surface \( S \) at an ordinary point \( O_1 \), up to and including the terms of the third order. This cubic curve is rational and its inflexions lie on the three tangents of Darboux through \( O_1 \). In this paper we give a projective characterization of such a curve, and hence a new derivation of the tangents of Darboux. By using the method employed in this characterization to the curve of intersection of the tangent plane of the surface at \( O_1 \) with \( S \), a simple characterization of the second edge of Green is found. Another application exhibits the correspondence of Moutard. Finally a new interpretation of the reciprocal of the projective normal is given in terms of the conditions of apolarity of a cubic form to a quartic form. The canonical tangent appears in a similar fashion.

Let \( S \) be referred to its asymptotic curves, and let the coordinates \((x^1, x^2, x^3, x^4)\) of the generic point \( O_1 \) of \( S \) be normalized so that they satisfy the system [2] of differential equations

\[
\begin{align*}
    x_{uu} &= \theta_u x_u + \beta x_v + px, \\
    x_{vv} &= \gamma x_u + \theta_v x_v + qx, \quad \theta = \log R.
\end{align*}
\]

(1.1)

The line \( l_1 \) joining \( O_1 \) to \( O_4 \), whose coordinates are \( x_{u_1} \), is the R-conjugate line, and the line \( l_2 \) determined by \( O_2, O_3 \), whose respective coordinates are \( x_{u_2}, x_{v_2} \), is the R-harmonic line.

If we define the local coordinates \((x_1, x_2, x_3, x_4)\) with respect to

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\(^1\) Numbers in brackets refer to the references cited at the end of the paper.
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$O_1O_2O_3O_4$ of a point $X$ by the expression

$$X^i = x_1x^i + x_2x^i + x_3x^i + x_4x^i,$$

and local nonhomogeneous coordinates $(x, y, z)$ by $x = x_2/x_1$, $y = x_3/x_1$, $z = x_4/x_1$, the power series expansion [4] of $S$ at $O_1$ is

$$(1.2) \quad z = xy - \frac{1}{3}(\beta x^3 + \gamma y^3) + \frac{1}{12}F_4(x, y) + \cdots,$$

wherein

$$(1.3) \quad F_4(x, y) = (2\beta_0 - \beta_0)x^4 - 4(\beta_0 + \beta_0)x^2y - 6\theta_0x^2y^2$$

$$- 4(\gamma_0 + \gamma_0)xy^2 + (2\gamma_0 - \gamma_0)y^4.$$

2. Characteristic points of a plane curve. Let the triangle of reference $O_1O_2O_3$ to which a plane curve $C$ is referred be covariant to the curve or to a surface to which $C$ bears some geometrical relation. Let the homogeneous coordinates of a point with respect to this triangle be $(x_1, x_3, x_3)$, the nonhomogeneous coordinates being defined by the expressions $x = x_2/x_1$, $y = x_3/x_1$. The line $y = 0$ being chosen as the tangent to $C$ at $O_1$, the power series expansion [4] of $C$ at $O_1$ is

$$(2.1) \quad y = a_2x^2 + a_3x^3 + a_4x^4 \cdots.$$

Consider at $O_2(0, 0, 1)$ the involution whose double lines are $O_1O_3$, $O_2O_3$. Corresponding lines of this involution intersect $C$ in points $P_1(x, y)$, $P_3(-x, y')$, $y' = a_2x^2 - a_3x^3 + a_4x^4 - \cdots$. The line $P_1P_3$ intersects the tangent to $C$ at $O_1$ in a point whose limit $T$ as $P_1$ approaches $O_1$ along $C$ has coordinates

$$(2.2) \quad x_1 = a_3, \quad x_2 = -a_2, \quad x_3 = 0.$$

We shall call the point $T$ with coordinates (2.2) the characteristic point of the second order of $C$ at $O_1$ relative to $O_1O_2O_3$.

Let $O_2(\rho, 1, 0)$ be an arbitrary point on the tangent to $C$ at $O_1$, but distinct from $O_1$. The transformation from the triangle $O_1O_2O_3$ to $O_1O_2O_3$ is

$$(2.3) \quad x = \frac{Ax'}{1 + \rho A x'}, \quad y = \frac{By'}{1 + \rho A x'}.$$

Under the transformation (2.3), the equation of $C$ may be written in the form

$$y' = a_2' x'^2 + a_3' x'^3 + \cdots,$$

wherein
\[ a'_2 = A^2 a_2 / B, \quad a'_3 = A^3 (a_3 - p a_2) / B. \]

Hence the characteristic point of \( C \) relative to \( O_1 O_3 O_2 \) has coordinates
\[ x_1 = (a_3 - 2 p a_2), \quad x_2 = - a_2, \quad x_3 = 0 \]
referred to \( O_1 O_3 O_2 \).

More generally let the equation of \( C \) have the form
\[ y = a_k x^k + a_{k+1} x^{k+1} + \cdots, \quad k \geq 2. \]

Consider through \( O_3 \) two lines forming with \( O_1 O_3, O_2 O_3 \) the constant cross ratio \( l, l \) being one of the \( k \)th roots of unity, but \( l \neq 1 \). These lines intersect \( C \) in two points \( P_1, P_2 \) determining a line which intersects the tangent to \( C \) at \( O_1 \) in a point whose limit as \( P_1 \) approaches \( O_1 \) has coordinates
\[ x_1 = a_{k+1}, \quad x_2 = - a_k, \quad x_3 = 0. \]

We shall call the point \( T \) whose coordinates are (2.5) the characteristic point of the \( k \)th order of \( C \) relative to \( O_1 O_3 O_2 \).

3. The characteristic curve of \( S \). Let us consider the section \( C_x \) of the surface \( S \) by a plane \( \pi \) through the \( R \)-conjugate line \( l_1 \). Let \( \pi \) intersect the \( R \)-harmonic line \( l_2 \) in \( O_3 \). The local coordinates of \( O_3 \) are of the form \( (0, \lambda, \mu, 0) \), and the local coordinates of any point \( Q \) on \( O_1 O_3 \) are \( (1, \lambda \xi, \mu \xi, 0) \). The equation of \( C_x \) referred to \( O_1 O_4 \) in nonhomogeneous coordinates \( (\xi, z) \) is
\[ z = \lambda \mu \xi^2 - \frac{1}{3} (\beta \lambda^3 + \gamma \mu^3) \xi^3 + \frac{1}{12} F_4(\lambda, \mu) \xi^4 + \cdots. \]

From (2.2) the characteristic point \( T_x \) of \( C_x \) relative to \( O_1 O_4 \) has coordinates
\[ \xi = 3 \lambda \mu / (\beta \lambda^3 + \gamma \mu^3), \quad z = 0, \]
referred to \( O_1 O_4 \), and coordinates
\[ x = 3 \lambda^2 \mu / (\beta \lambda^3 + \gamma \mu^3), \quad y = 3 \lambda \mu^2 (\beta \lambda^3 + \gamma \mu^3), \quad z = 0 \]
referred to \( O_1 O_3 O_4 O_4 \). The locus of \( T_x \) as \( \pi \) rotates about \( l_1 \) is the covariant rational cubic curve \( \Gamma_3 \) whose equation is
\[ 3 x y - (\beta x^3 + \gamma y^3) = 0, \quad z = 0. \]

We shall call this cubic the characteristic curve of \( S \) relative to \( l_1, l_2 \). The nodal tangents of \( \Gamma_3 \) are of course the asymptotic tangents of \( S \) at \( O_1 \), and the inflexions lie on the tangents of Darboux. The \( R \)-harmonic line
is the flex-ray of $\Gamma_3$.

From (3.3) it follows that the only sections of $S$ through the $R$-conjugate line whose characteristic points relative to $O_1O_3O_2$ lie on the $R$-harmonic line are those through the tangents of Darboux.

Another characterization of the cubic $\Gamma_3$ may be found in the following manner. The osculating conic of the section $C_\tau$ has the equation [4]

$$
\lambda^3 \mu^3 \left( z - \lambda \mu \xi^2 \right) + \frac{1}{3} \lambda^3 \mu^3 \left( \beta \lambda^3 + \gamma \mu^3 \right) \xi^2
$$

$$
+ \left[ \frac{1}{9} \left( \beta \lambda^3 + \gamma \mu^3 \right)^2 - \frac{1}{12} F_4(\lambda, \mu) \right] \xi^2 = 0.
$$

(3.5)

The pole of $R$-conjugate line with respect to this conic is the point $T'_\tau$ with coordinates

$$
\xi = -3 \lambda \mu / (\beta \lambda^3 + \gamma \mu^3), \quad \tau = 0.
$$

The harmonic conjugate of $T'_\tau$ with respect to $O_1O_3$ is the point $T_\tau$ defined by (3.2). Incidentally the locus of $T'_\tau$ is the cubic $\Gamma'_3$,

$$
3xy + \beta x^3 + \gamma y^3 = 0.
$$

The tangents of Darboux are thus again exhibited by means of $\Gamma'_3$.

Finally we may readily show that the polar line of the conic (3.5) intersects $O_4O_2$ in a point whose locus as $\pi$ varies is a rational curve of order seven which intersects the $R$-harmonic line at its intersections with the tangents of Darboux.

4. The edges of Green. The expansions [4] of the two branches of the curve of intersection of $S$ at $O_1$ with its tangent plane are

$$
y = \frac{1}{3} \beta x^2 - \frac{1}{12} (2\beta \theta_u - \beta_u)^3 + \cdots, \quad z = 0;
$$

$$
x = \frac{1}{3} \gamma y^2 - \frac{1}{12} (2\gamma \theta_v - \gamma_v)^3 + \cdots, \quad z = 0.
$$

(4.1)

The characteristic point $T_u$ of the first of (4.1) relative to $O_1O_3O_2$ has coordinates

$$
x_1 = \frac{1}{4} \left( 2\theta_u - \frac{\beta_u}{\beta} \right), \quad x_2 = 1, \quad x_3 = x_4 = 0,
$$

(4.2)

and the characteristic point $T_v$ of the second relative to $O_1O_3O_2$ has coordinates
The line joining the harmonic conjugates of $T_u$ and $T_v$ with respect to $O_1O_2$ and $O_3O_4$ respectively is Green's edge of the second kind.

This edge of Green may be characterized in another way. The section of $S$ by the plane through the $R$-conjugate line and the tangent to the asymptotic curve $v=\text{const.}$ has the equation

$$z = -\frac{1}{3} \beta x^3 + \frac{1}{12} (2\beta\theta_u - \beta_u) x^4 + \cdots.$$  

The characteristic point of the third order of the curve (4.4) relative to $O_1O_4O_2$, is found from (2.5) to have coordinates given by (4.2); by interchanging the roles of the asymptotic tangents the point (4.3) is characterized. The second edge of Green is therefore given another characterization.

Consider on the tangent to the section (3.1) $C_r$ of $S$ the point $O_r (\rho, 2\lambda, 2\mu, 0)$. From (2.4) we find readily that the characteristic point $T$ of $C_r$ relative to $O_1O_4O_2'$ has coordinates

$$x_1 = \rho \lambda \mu + \frac{1}{3} (\beta \lambda^3 + \gamma \mu^3), \quad x_2 = \lambda^2 \mu, \quad x_3 = \lambda \mu^2, \quad x_4 = 0.$$  

The point $P_r$ on the tangent to $C_r$ at $O_1$ which with $O_1$ separates $O_r$ and $O_r$ harmonically has coordinates $(\rho, \lambda, \mu, 0)$. Equations (4.4) therefore represent a cubic transformation of $P_r$ into the characteristic point of $C_r$ relative to $O_1O_4O_2'$. The polar plane of the point (4.5) with respect to any quadric of Darboux,

$$x_2x_3 - x_1x_4 + k x_4^2 = 0,$$

has coordinates

$$\xi_1 = 0, \quad \xi_2 = \lambda \mu^2, \quad \xi_3 = \lambda^2 \mu, \quad \xi_4 = -\rho \lambda \mu - \frac{1}{3} (\beta \lambda^3 + \gamma \mu^3).$$  

The correspondence (4.6) between $P_r$ and the polar plane of the characteristic point of $C_r$ relative to $O_1O_4O_2'$ is the correspondence of Moutard ($k = -1/3$). We have previously [3] given a different derivation of this correspondence.

5. The projective normal. The surface $S'$ whose equation is

$$z = xy - \frac{1}{3} (\beta x^3 + \gamma y^3).$$
has a unode at $O_4$, the plane $O_2O_3O_4$ as uniplane, and has contact of the third order with $S$ at $O_1$; hence $S'$ is completely determined. The projection on their common tangent plane at $O_1$ of the curve of intersection of $S$ and $S'$ has a quadruple point at $O_1$, the quadruple tangents being given by

\begin{equation}
F_4(x, y) = 0
\end{equation}

where $F_4(x, y)$ is defined by (1.3). The lines (5.2) intersect the R-harmonic line in four points $F_i$, and the Segre tangents intersect this line in three points $S_i$. It is easy to verify that \textbf{the points $S_i$ are apolar to $F_i$ if and only if the R-harmonic line is the reciprocal of the projective normal.} The projective normal is therefore geometrically determined by reciprocation with respect to the quadrics of Darboux.

Finally let the lines $l_1, l_2$ be the projective normal and its reciprocal; then it readily follows that the polar of the form $\beta x^3 + \gamma y^3$ with respect to $F_4(x, y)$ is

\begin{equation}
\phi x - \psi y
\end{equation}

wherein $\phi = \partial \log (\beta \gamma^3)/\partial u$, $\psi = \partial \log (\beta \gamma^3)/\partial v$. The form (5.3) equated to zero is seen to be \textit{the equation of the canonical tangent}.

\textbf{References}


4. E. P. Lane, \textit{A treatise on projective differential geometry}, The University of Chicago Press, 1942.