A NOTE ON LOCAL CONNECTIVITY

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Two neighborhoods of a point are involved in the definition\(^1\) of local connectivity: a space \(T\) is \(p\)-LC at a point \(x\) if every neighborhood \(U\) of \(x\) contains a neighborhood \(V\) of \(x\) such that any continuous \(p\)-sphere in \(V\) bounds a continuous \((p+1)\)-cell in \(U\). \(T\) is \(p\)-LC if it is \(p\)-LC at every point, and it is \(LC^n\) if it is \(p\)-LC for \(0 \leq p \leq n\).

For the case \(p = 0\), it is well known that there is an equivalent definition: a space \(T\) is \(0\)-LC if every point has arbitrarily small neighborhoods \(V\) such that any continuous 0-sphere in \(V\) bounds a continuous 1-cell in \(V\). But for \(p > 0\), Borsuk and Mazurkiewicz have shown by an example\(^2\) that these two definitions are not equivalent.

Hence the question arises as to the relative size of \(V\) with respect to \(U\) in the first definition. At first glance, the Borsuk-Mazurkiewicz example would seem to indicate that \(V\) must be considerably smaller than \(U\). This, however, is not the case.

**Theorem.** If a space \(T\) is \(LC^n\), then each point of \(T\) has arbitrarily small neighborhoods \(V\) such that any continuous \(p\)-sphere, \(0 \leq p \leq n\), in \(V\) bounds a continuous \((p+1)\)-cell in \(V\).

**Proof.** Let \(U\) be a neighborhood of a point \(x\) of \(T\) such that any continuous 0-sphere in \(U\) bounds a continuous 1-cell in \(U\). Let \(A\) be the class of all neighborhoods \(V\) of \(x\) such that any continuous \(p\)-sphere, \(0 < p \leq n\), in \(V\) bounds a continuous \((p+1)\)-cell in \(U\). Since \(T\) is \(LC^n\), \(A\) is not vacuous. Order the elements of \(A\) by inclusion. Since the continuous image of a sphere is a compact set, the union of the elements of any simply ordered subset of \(A\) is again an element of \(A\). Hence, by Zorn's lemma, \(A\) contains a maximal element, \(V_0\).

We assert that \(V_0 \supseteq U\). If not, let \(y\) be a point of the open set \(U - V_0\), and let \(W\) be a neighborhood of \(y\), \(W \subseteq U - V_0\), such that any continuous \(p\)-sphere, \(0 < p \leq n\), in \(W\) bounds a continuous \((p+1)\)-cell in \(U\). Since \(p > 0\), any continuous \(p\)-sphere in \(V_0 \cup W\) is either in \(V_0\) or in \(W\), so \(V_0 \cup W\) is an element of \(A\), which contradicts the maximality of \(V_0\).

Received by the editors May 2, 1947.


Now, by the original choice of \( U, V \) has the property required by the theorem.

We remark that the same proof holds, with trivial modifications, for homology local connectivity.

It is an open question whether the neighborhood \( V \) can be required to be connected.

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UNCONDITIONAL CONVERGENCE IN BANACH SPACES

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Introduction. This note investigates an apparent generalization of unconditionally convergent series \( \sum x_i \) in weakly complete Banach spaces. A series of elements with \( x_i \) in \( E \) is said to be unconditionally convergent if for every variation of sign \( e_i = \pm 1 \), \( \sum e_i x_i \) is convergent. This formulation of the definition of unconditional convergence is equivalent to that given by Orlicz[4]. We call \( \sum x_i \) unconditionally summable if there exists a finite row Toeplitz matrix \( (b_{ik}) \) such that for every variation of sign \( \sigma_i = \sum_{i=1}^{n} b_{ik} \sum_{i=1}^{k} e_i x_i \) converges. The fact that unconditional summability implies unconditional convergence is established in this note. Finally, applications to orthogonal functions are presented.

Preliminary lemmas. In what follows, \( b_{ik} \) will denote an arbitrary finite row Toeplitz matrix.

**Lemma 1.** If \( S_n(\theta) = \sum_{i=1}^{n} a_{r_i}(\theta) \) converges to an essentially bounded function \( f(\theta) \), then \( \left| \sum_{i=1}^{n} a_{r_i}(\theta) \right| \leq c \) almost everywhere. (\( r_n(\theta) \) denote the Rademacher functions.)

**Proof.** This is an immediate consequence of the result that

\[
\left( \int_0^1 \left( \max_{1 \leq n \leq m} \left| \sum_{i=1}^{n} a_{r_i}(\theta) \right| \right)^p d\theta \right)^{1/p} \leq C \left( \int_0^1 |S_n(\theta)|^p d\theta \right)^{1/p}, \quad 1 \leq p \leq \infty,
\]

Received by the editors April 30, 1947.

1 Numbers in brackets refer to the references cited at the end of the paper.