NOTE ON THE LOCATION OF THE CRITICAL POINTS
OF HARMONIC FUNCTIONS

J. L. WALSH

By a limiting process, a theorem recently proved by the writer can be generalized, and yields a new result with interesting applications which we wish to record here. We take as point of departure the following theorem.

**Theorem 1.** Let the region $R$ of the extended $(x, y)$-plane be bounded by a finite number of mutually disjoint Jordan curves $C_0, C_1, C_2, \ldots, C_n$. Let the function $u(x, y)$ be harmonic in $R$, continuous in the corresponding closed region, equal to zero on $C_0$ and to unity on $C_1, C_2, \ldots, C_n$. Denote by $R_0$ the region bounded by $C_0$ containing the curves $C_1, C_2, \ldots, C_n$ in its interior; define noneuclidean straight lines in $R_0$ as the images of arcs of circles orthogonal to the unit circle, when $R_0$ is mapped conformally onto the interior of the unit circle.

If $\Pi$ is any non-euclidean convex region in $R_0$ which contains all the curves $C_1, C_2, \ldots, C_n$, then $\Pi$ also contains all critical points of $u(x, y)$ in $R$.

We extend Theorem 1 by admitting arcs of $C_0$ on which $u(x, y)$ is prescribed to take the value unity, and also by admitting the intersection of curves $C_1, C_2, \ldots, C_n$ with $C_0$:

**Theorem 2.** Let the region $R$ be bounded by the whole or part of the Jordan curve $C_0$, and by mutually disjoint Jordan arcs or curves $C_1, C_2, \ldots, C_n$ in the closed interior of $C_0$; some or all of the latter arcs or curves may have points in common with $C_0$. Let a finite number of arcs $\alpha_1, \alpha_2, \ldots, \alpha_m$ of $C_0$ belong to the boundary of $R$ and be mutually disjoint. Let the function $u(x, y)$ be harmonic and bounded in $R$, and take continuously the boundary values unity on $C_1, C_2, \ldots, C_n, \alpha_1, \alpha_2, \ldots, \alpha_m$ and zero in the remaining boundary points of $R$, except that in points common to $C_n$ and $C_1+C_2+\cdots+C_n$ and in end points of the $\alpha_i$ no continuous boundary value is required. Denote by $R_0$ the region bounded by $C_0$ containing $R$, and define noneuclidean straight lines in $R_0$ by mapping $R_0$ onto the interior of a circle. If $\Pi$ is any closed region in the closure of $R_0$ which is non-euclidean convex and which contains $C_1+C_2+\cdots+C_n+\alpha_1+\alpha_2+\cdots+\alpha_m$, then $\Pi$ contains all critical points of $u(x, y)$ in $R$.

Presented to the Society, April 26, 1947; received by the editors April 21, 1947.

Theorem 2 may be proved by mapping $R_0$ onto the interior of the unit circle; we retain the original notation. The region $R$ can be approximated by a region $R'$ bounded by $C_0$ and by Jordan curves $C_1', C_2', \cdots, C_n'$, $\alpha_1', \alpha_2', \cdots, \alpha_m'$ in $R_0$ which are mutually disjoint and disjoint with $C_0$ and which respectively approximate $C_1, C_2, \cdots, C_n, \alpha_1, \alpha_2, \cdots, \alpha_m$. Let the function $u'(x, y)$ be harmonic in $R'$, continuous in the corresponding closed region, zero on $C_0$ and unity elsewhere on the boundary of $R'$. Then as $R'$ suitably approaches $R$, the variable function $u'(x, y)$ approaches $u(x, y)$ throughout $R$, uniformly on any closed set interior to $R$; we omit the proof. Any critical point of $u(x, y)$ interior to $R$ is a limit point of critical points of the variable function $u'(x, y)$, so Theorem 2 follows from Theorem 1.

A further general result has recently been established for the case $n = 0$, which constructs $\Pi$ in $R_0$ not by joining the ends of each arc of $C_0$ in the complement of the set $\alpha_j$ by a non-euclidean line but by similarly joining the ends of each double arc composed of an $\alpha_j$ plus one of the adjoining arcs of $C_0$ complementary to the set $\alpha_1 + \alpha_2 + \cdots + \alpha_m$. It is still true (we shall refer to this result as Theorem 3) that $\Pi$ contains all critical points in $R$ of the corresponding harmonic function $u(x, y)$.

Theorem 3 is more powerful than Theorem 2 for the case $n = 0$, but requires for its application essentially the use of a specific conformal map, and the latter quality may be an advantage or a disadvantage. It is an indication of the power of Theorem 2 that in the application of it to a given configuration, with or without the auxiliary use of conformal mapping, there may obviously be some arbitrariness in the notation, especially as to what shall be chosen as the region $R_0$. So far as convenience is concerned, it is desirable to choose simple configurations, where the totality or useful subset of non-euclidean lines is easily determined. It is also well to choose $R_0$ in such a way that the point set $C_1 + \cdots + C_n + \alpha_1 + \cdots + \alpha_m$ is as nearly non-euclidean convex as possible. But if the aim is precision, the larger $R_0$ the better, as we proceed to indicate in a special but typical case.

In Theorem 2, let $C_0$ be the unit circle in the $z$-plane, $n = 2, m = 0$, with $C_1$ and $C_2$ mutually disjoint Jordan arcs whose end points lie on $C_0$ and whose interior points lie interior to $C_0$. Let the subregion $R$ of the interior of $C_0$ be bounded by $C_1, C_2$, and two appropriate arcs of $C_0$. In the actual application of Theorem 2, we can choose $R_0$ as the interior of $C_0$, or as the region $R_1$ containing $R$ bounded by $C_1$.

---

and a suitable arc of $C_0$, or as the region $R_2$ containing $R$ bounded by $C_2$ and a suitable arc of $C_0$, or as $R$. We now show as a general indication but without a complete rigorous proof that among these choices the most precise results are obtained by choosing $R_0$ as the interior of $C_0$.

Map (for instance) the region $R$ onto the interior of the unit circle in the $w$-plane. Let $\alpha_z$ be an arbitrary arc of $C_0$ belonging to the boundary of $R$, which corresponds to the arc $\alpha_w$ in the $w$-plane. Let $\alpha'_z$ be the circular arc having the same end points as $\alpha_z$, orthogonal to $\alpha_z$, and whose interior points lie interior to $C_0$; we assume $\alpha'_z$ to lie in the closure of $R$. Let $\alpha'_w$ be the circular arc having the same end points as $\alpha_w$, orthogonal to $\alpha_w$, and whose interior points lie in $|w| < 1$. The arcs $\alpha'_z$ and $\alpha'_w$ determine the respective non-Euclidean geometries in the $z$-plane and $w$-plane, and it follows from a general theorem due to R. Nevanlinna\(^3\) that the region bounded by $\alpha_z$ and $\alpha'_z$ contains every point of the image of the region bounded by $\alpha_w$ and $\alpha'_w$. Corresponding to every arc $\alpha_z$ belonging to the boundary of $R$ and on which the prescribed boundary value of $u(x, y)$ is zero, and to the adjacent arc $\alpha'_z$, with no point of $C_1$ or $C_2$ in the lens-shaped region between $\alpha_z$ and $\alpha'_z$ there exists in the $w$-plane an arc $\alpha_w$ whose end points correspond to those of $\alpha_z$ under the conformal map, such that the interior points of the arc $\alpha'_w$ lie interior to the lens-shaped region bounded by $\alpha_w$ and the image of $\alpha'_z$. It follows that if we neglect arcs $\alpha'_z$ that cut $C_1$ or $C_2$ in $R$, then in this particular case Theorem 2 can be more favorably applied by choosing $R_0$ as the interior of $C_0$, that is to say, as large as possible.

The remark just made is of fairly general application. Moreover, in the specific case used, the interior of the given $C_0$ may be enlarged, without altering $R$ or $u(x, y)$, by adding to $R_0$ regions adjacent to the arcs $A$ of the given $C_0$ bounded by the end points of $C_1$ and $C_2$, the arcs $A$ not being part of the boundary of $R$. Indeed, we may even adjoin an infinitely many sheeted logarithmic Riemann surface along each arc $A$; this is equivalent to mapping onto the interior of the unit circle the region $R$ plus auxiliary regions, so that with the omission of two exceptional points the circumference of the unit circle corresponds only to that part of the boundary of $R$ on which the prescribed boundary value of $u(x, y)$ is zero. The image of $C_1$ (and likewise of $C_2$) under this map is a Jordan curve which except for a single point lies interior to the unit circle.

Still another instructive kind of conformal map can be used, namely to map $R + C_1 + C_2$ onto the interior of the unit circle in such a way

---

\(^3\) Eindeutige analytische Funktionen, Berlin, 1936, p. 51.
that $C_1$ and $C_2$ correspond to radial slits, while the part of the boundary of $R$ on which $u(x, y)$ has the prescribed boundary value zero corresponds to the whole circumference less two points. Here the region II of Theorem 2 may degenerate to a line segment.

Another indication of the power of Theorem 2 is the following. Let $C_0$ be the unit circle, $n = 1$ with $C_1$ a concentric circle of radius $r_1 < 1$; let an arc $\alpha$ (not the whole circle) of $C_0$ contain all the arcs $\alpha_j$. By a conformal map of the universal covering surface of $R$ onto the unit circle and application of Theorem 3 extended to the case of an infinite number of arcs, $\alpha_j$, it follows (loc. cit. footnote 2) that in the original plane no critical points of $u(x, y)$ lie in the annulus $r_1 < r < r_1^{1/2}$; a second annulus $r_2 < r < 1$ free from critical points can also be determined by this method. By Theorem 2, any circle cutting $C_0$ orthogonally in two points of the complement of $\alpha$ and containing in its interior no point of $\alpha$ or of $C_1$ contains in its interior no critical point of $u(x, y)$. In all, these conclusions may leave only a very small subregion of $R$ as the portion in which the critical points of $u(x, y)$ lie.

We continue with a generalization of this result, a further application of Theorem 2:

**Theorem 4.** Let $R$ be a region bounded by the whole of the Jordan curve $C_0$, by the whole or part of the Jordan curve $C_1$ disjoint from $C_0$, and by mutually disjoint Jordan arcs or curves $C_2, C_3, \ldots, C_n$ in the closed interior of the annulus $R_0$ bounded by $C_0$ and $C_1$; some or all of the latter arcs and curves are permitted to have points in common with $C_1$, but none with $C_0$. Let a finite number of arcs $\beta_1, \beta_2, \ldots, \beta_m$ of $C_1$ be part of the boundary of $R$ and mutually disjoint. Let the function $u(x, y)$ be harmonic and bounded in $R$, and take continuously the boundary value zero on $C_0 + \beta_1 + \beta_2 + \cdots + \beta_m$ and unity in the remaining boundary points of $R$, except that in points common to the $\beta_j$ and $C_2 + \cdots + C_n$ and in end points of the $\beta_j$, no continuous boundary value is required.

If $\omega(z, C_0, R_0)$ denotes the harmonic measure of $C_0$ in the point $z$ with respect to the annular region $R_0$, then for constant $\mu$ the largest region $\omega(z, C_0, R_0) > \mu \geq 1/2$ which contains no points of $C_2 + \cdots + C_n$ contains no critical points of $u(x, y)$.

Theorem 4 is proved by mapping onto the unit circle the universal covering surface of $R_0$, and by applying a slight generalization of Theorem 2. We omit the proof.

We turn now to a generalization of Theorems 2 and 4, in a more general situation. Let $R$ be a region bounded by the whole or part of
the mutually disjoint Jordan curves, $C_1, C_2, \ldots, C_k$ (which together bound a region $R_0$) and by mutually disjoint Jordan arcs or curves $C_{k+1}, \ldots, C_n$ in the closure of $R_0$; some or all of the latter arcs or curves may have points in common with $C_1 + C_2 + \cdots + C_k$. Let a finite number of arcs $\alpha_1, \alpha_2, \ldots, \alpha_m$ of $C_1 + C_2 + \cdots + C_k$ belong to the boundary of $R$ and be mutually disjoint. Let the function $u(x, y)$ be harmonic and bounded in $R$, and take continuously the boundary values unity on $C_{k+1} + \cdots + C_n + \alpha_1 + \cdots + \alpha_m$ and zero in the remaining boundary points of $R$, except that in points common to $C_1 + \cdots + C_k$ and $C_{k+1} + \cdots + C_n$ and in end points of the $\alpha_j$, no continuous boundary value is required. In studying the location of the critical points of $u(x, y)$, in order to apply Theorem 2 (in generalized form), it is natural to map onto the interior of the unit circle the universal covering surface of $R_0$. Any non-euclidean convex region in the unit circle containing all image points of the set $C_{k+1} + \cdots + C_n + \alpha_1 + \cdots + \alpha_m$ contains all critical points of the transform of $u(x, y)$. But here we have a large choice; we may change the notation so that any subset of the arcs or curves $C_{k+1}, \ldots, C_n$ belongs to the set $C_1, C_2, \ldots, C_k$; each choice of the subset yields a new region $R_0$, a new conformal map, a new non-euclidean geometry, a new application of Theorem 2 (generalized), and a new conclusion.

Throughout the present note we have studied in detail harmonic functions which for a simply connected region $R_0$ take on the values zero (on arcs of the boundary of $R_0$) and unity (on arcs of the boundary or curves in $R_0$). By the same methods one can also study harmonic functions which take on the values zero (on arcs of the boundary of $R_0$), unity (on arcs of the boundary or curves in $R_0$), and minus unity (on arcs of the boundary or curves in $R_0$); the results generalize those previously obtained by the writer (loc. cit.) and can be still further generalized to regions of higher connectivity by a conformal map of the universal covering surfaces of such regions.

Harvard University