SOME LIMIT THEOREMS
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1. Introduction. It is a classical result in the theory of trigonometric series that if
\[ c_n \cos nx + d_n \sin nx \to 0 \quad (n \to \infty) \]
for all (real) \( x \) on a set of positive measure, then
\[ c_n \to 0, \quad d_n \to 0. \]
Cantor proved this for the case that set \( \{x\} \) is an interval, and Lebesgue established the result for a set of measure zero. A short proof is given by Hardy and Rogosinski.\(^1\)

The following related result was proved and used by Szász.\(^2\) If
\[ a_n \sin nx + b_n \sin (n+1)x \to 0 \]
on a (real) set \( \{x\} \) of positive measure, then
\[ a_n \to 0, \quad b_n \to 0. \]

Relations (1.1) and (1.3) can be put into complex form. For example, (1.1) becomes
\[ a_n \exp \{nx\} + b_n \exp \{-nx\} \to 0, \]
with the conclusion that
\[ a_n \to 0, \quad b_n \to 0. \]
Here \( \exp \{u\} \) is defined by
\[ \exp \{u\} = e^{iu} \quad (i = (-1)^{1/2}). \]

Our purpose in the present work is to extend the conclusions of the above-mentioned results to combinations more general than (1.3), (1.5). Thus in §2 we go from two terms to \( k \) terms and generalize the exponents; in §3 the coefficients of the exponentials are permitted

\(^{1}\) Hardy and Rogosinski, *Fourier series* (Cambridge Tracts in Mathematics and Mathematical Physics, no. 38), Theorem 92, p. 84.

\(^{2}\) Otto Szász, *On Lebesgue summability and its generalization to integrals*, Amer. J. Math. vol. 67 (1945) pp. 389–396, especially Lemma 2, p. 395. Dr. Szász has informed me that, with the intention of using it in work on trigonometric series, he has proved (but not published) a generalization of (1.3), namely where the left side of (1.3) is replaced by the expression \( \sum_{n=0}^{N} a_n e^{iu} \).
2. **One-dimensional case.** In the following sections we suppose without further mention that all variables $x, y, \cdots$ are real.

**Lemma 2.1.** Let $\{u_n\}$ be a real sequence that does not have zero as its limit. The relation

$$\lim_{n \to \infty} \exp \{u_n x\} = 1$$

cannot hold on a set of positive measure.$^8$

Suppose the lemma is false, so that there is a set $\mathcal{J}$ of positive measure on which (2.1) is satisfied. We may suppose that $\mathcal{J}$ is bounded. It is no restriction to assume that zero is not a limit point of $\{u_n\}$; for there exists an infinite subsequence of $\{u_n\}$ for which zero is not a limit point, and we may remove all $u_n$'s not in this subsequence.

Suppose $\{u_n\}$ contains a bounded subsequence $\{u_{n_j}\}$; then from $\{u_{n_j}\}$ a further sequence can be chosen for which a limit exists. This limit, say $L$, cannot be zero, so from $\exp \{Lx\} = 1$ for $x$ in $\mathcal{J}$ we conclude that $\mathcal{J}$ is at most a denumerable set, contrary to the assumption that $\mathcal{J}$ is of positive measure.

Now suppose that $\{u_n\}$ contains no bounded subsequence, so that $|u_n| \to \infty$. By Egoroff's theorem there is a subset $\mathcal{J}_1$ of $\mathcal{J}$, of positive measure, on which (2.1), that is,

$$\cos u_n x + i \sin u_n x \to 1,$$

holds uniformly. Consequently, since $1 + \cos u_n x$ is uniformly bounded,

$$\cos^2 u_n x - 1 \to 0 \quad \text{(uniformly on $\mathcal{J}_1$)}.$$

Integrating over $\mathcal{J}_1$:

$$\int_{\mathcal{J}_1} \cos^2 u_n x dx \to \int_{\mathcal{J}_1} dx = m(\mathcal{J}_1).$$

Now there exists an open set $\mathcal{Q}$, consisting of a finite number of nonoverlapping intervals, say $(a_j, b_j)$, $j = 1, \cdots, \rho$, with the following two properties: (i) $\mathcal{Q}$ contains $\mathcal{J}_1$; (ii) $m(\mathcal{J}_1) \leq m(\mathcal{Q}) < 3m(\mathcal{J}_1)/2$. Hence,

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$^8$ As originally stated and proved, this and the other results of the paper used the condition *interval* everywhere in place of *set of positive measure*. We owe to Dr. Szász the suggestion to generalize to the case of positive measure, and also to extend the results to more than one variable (see §4).
\[
\int_{\mathcal{F}_1} \cos^2 u_n x dx \leq \int_{\mathcal{Q}} \cos^2 u_n x dx = 2^{-1} \int_{\mathcal{Q}} (1 + \cos 2u_n x) dx
\]
\[
= 2^{-1} m(\mathcal{Q}) + 2^{-1} \sum_{j=1}^{p} \left( \frac{\sin 2u_n x}{2u_n} \right)_{a_j};
\]
so for all \( n \) sufficiently large,
\[
(2.3) \quad \int_{\mathcal{F}_1} \cos^2 u_n x dx < \frac{3}{4} m(\mathcal{F}_1).
\]
But this contradicts (2.2), so the lemma cannot be false. 4

**Lemma 2.2.** Let \( \{t_{s,n}\}, s = 1, \cdots, k, \) be real sequences with the property that none of the sequences \( \{t_{s,n}\}, \{t_{s,n} - t_{p,n}\} \) (\( s \neq p \)) has zero as a limit point. If real or complex constants \( \{A_{s,n}\} \) exist such that
\[
(2.4) \quad \sum_{s=1}^{k} A_{s,n} \exp \{t_{s,n} x\} - 1 \to 0
\]
for all \( x \) on a set \( \mathcal{E} \) of positive measure, then
\[
(2.5) \quad A_{s,n} \to 0 \quad (s = 1, \cdots, k).
\]
If \( k = 1 \) the lemma is true in virtue of Lemma 2.1. Assume it true for the case \( k - 1 \); we shall then prove it for \( k \) by an induction argument, and this will establish the truth of Lemma 2.2.

It is no restriction to suppose that \( \mathcal{E} \) is a bounded set. \( \mathcal{E} \) contains a point \( x_1 \) with the property that every interval containing \( x_1 \) in its interior meets \( \mathcal{E} \) in a set of positive measure. 5 For suppose not. Then about each \( x \) in \( \mathcal{E} \) exists an interval \( I_x \), with \( x \) in its interior, such that \( \mathcal{E} \cdot I_x \) is of measure zero. Let \( x_1 \) be in \( \mathcal{E} \), and let \( I_{x_1} \) be the largest associated interval. It is clear that there is a largest interval. If \( x_2 \) in \( \mathcal{E} \) is not in \( I_{x_1} \), then it too has a largest associated interval \( I_{x_2} \), and \( I_{x_1} \), \( I_{x_2} \) do not meet. It is now a straightforward argument to show that \( \mathcal{E} \) is covered by at most a denumerable number of such intervals \( I_x \), thus establishing \( \mathcal{E} \) as a set of zero measure. This contradiction shows that a point such as the aforementioned \( x_1 \) exists.

Let \( x \) in (2.4) take on such a value \( x_1 \) and subtract from (2.4). There results the relation

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4 This contradiction does not preclude the possibility that the set of points for which (2.1) holds is non-measurable, but in this case the set cannot contain a subset of positive measure.

5 A much stronger conclusion as to the number of such points \( x_1 \) is of course possible, but we require only the above mild assertion.
for all \( y \) in \( \mathcal{E}_1 = \{ y = x - x_1, \text{ as } x \text{ ranges over } \mathcal{E} \} \). \( \mathcal{E}_1 \) and \( \mathcal{E} \) have the same (positive) measure, and the replacement of (2.4) and \( \mathcal{E} \) by (2.6) and \( \mathcal{E}_1 \) insures that the origin (\( y = 0 \)) is a point of \( \mathcal{E}_1 \) every neighborhood of which contains a subset of \( \mathcal{E}_1 \) of positive measure.

Let

\[
B_{s,n} = A_{s,n} \exp \{ t_{s,n} x_1 \}.
\]

Then

\[
\sum_{s=1}^{k} B_{s,n} [\exp \{ t_{s,n} y \} - 1] \to 0 \quad (y \text{ in } \mathcal{E}_1),
\]

and (2.5) is equivalent to

\[
B_{s,n} \to 0 \quad (s = 1, \cdots, k).
\]

Suppose the lemma is false for case \( k \). Then a value \( s \), say \( s = k \), exists such that

\[
|B_{k,n(j)}| > M;
\]

so on replacing \( \{ n \} \) by \( \{ n(j) \} \) in (2.8) and dividing by \( B_{k,n(j)} \), we have

\[
\sum_{s=1}^{k-1} C_{s,n(j)} [\exp \{ t_{s,n(j)} y \} - 1] \to 0 \quad (y \text{ in } \mathcal{E}_1).
\]

Here

\[
C_{s,n(j)} = \frac{B_{s,n(j)}}{B_{k,n(j)}}.
\]

Let \( y_1 \) be an arbitrary point of \( \mathcal{E}_1 \). The set of points \( \{ u \} \) defined by \( u = y - y_1 \) as \( y \) ranges over \( \mathcal{E}_1 \) will be denoted by \( \mathcal{E}_{y_1} \), and will be termed a translation set (relative to \( \mathcal{E}_1 \)). Clearly, \( m(\mathcal{E}_{y_1}) = m(\mathcal{E}_1) \).

Let \( \mathcal{Q} \) be an open set containing \( \mathcal{E}_1 \), with \( m(\mathcal{E}_1) < m(\mathcal{Q}) < 3m(\mathcal{E}_1)/2 \). There exists a set of nonoverlapping open intervals \( \mathcal{Q}_1 = I_1 + \cdots + I_r \), contained in \( \mathcal{Q} \), for which

\footnote{Actually, for every \( s = 1, \cdots, k \) the quantity \( B_{s,n} \) does not approach zero; for otherwise we can drop from sum (2.8) all terms for which \( B_{s,n} \to 0 \), and thus reduce (2.8) to \( k - 1 \) or less terms, in which case the lemma is true by our induction assumption.}
\[ m(\mathcal{E}_1) < m(\mathcal{Q}_1) < 3m(\mathcal{E}_1)/2, \]

and also such that
\[ m(\mathcal{Q}_1 \cdot \mathcal{E}_1) > 3m(\mathcal{E}_1)/4. \]

To each end of \( I_p \) (\( p = 1, \ldots, r \)) add an interval of length \( \Delta \), forming a new interval \( J_q \), where \( \Delta \) is chosen small enough so that on setting \( \mathcal{Q}_2 = J_1 + \cdots + J_q \) (\( q \leq r \) since some intervals may overlap and thus be combined) then \( m(\mathcal{Q}_2) < 3m(\mathcal{E}_1)/2. \)

Let \( \mathcal{K} \) be the subset of numbers \( y \) of \( \mathcal{E}_1 \) for which \( |y| < \Delta \). We know that \( \mathcal{K} \) is of positive measure. Moreover, for an arbitrary \( y \) in \( \mathcal{K} \),
\[ m(\mathcal{Q}_2 \cdot \mathcal{E}_y) > 3m(\mathcal{E}_1)/4, \quad m(\mathcal{Q}_2 \cdot \mathcal{E}_i) > 3m(\mathcal{E}_1)/4. \]

Since \( m(\mathcal{Q}_2) < 3m(\mathcal{E}_1)/2 \), it follows that
\[ m(\mathcal{E}_1 \cdot \mathcal{E}_y) > 0 \quad \text{(all } y \text{ in } \mathcal{K}). \]

We see from (2.10) and Lemma 2.1 that we cannot have \( C_{s,n(j)} \to 0 \) for all \( s = 1, \ldots, k-1 \). Hence there is an \( s \), say \( s = 1 \), for which \( C_{1,n(j)} \to 0 \) is false; and a subsequence \( \{m(j)\} \) of \( \{n(j)\} \), and a positive number \( K \), such that
\[ |C_{1,m(j)}| > K. \]

Now the relation
\[ \exp \{ (t_{1,m(j)} - t_{k,m(j)})y \} \to 1 \quad (j \to \infty) \]
cannot hold on a set of positive measure (Lemma 2.1); consequently, there is a point \( y_1 \) in \( \mathcal{K} \) such that (2.13) is false for \( z = y_1 \). Choose \( y = y_1 \) in (2.10) and subtract from (2.10):
\[ \exp \{ t_{k,n(j)}y_1 \} \left[ \exp \left\{ t_{k,n(j)}u \right\} - 1 \right] \]
\[ + \sum_{s=1}^{k-1} C_{s,n(j)} \exp \left\{ t_{s,n(j)}y_1 \right\} \left[ \exp \left\{ t_{s,n(j)}u \right\} - 1 \right] \to 0, \]
where \( u = y - y_1 \) (\( y \) in \( \mathcal{E}_1 \)), so that \( u \) ranges over the translation set \( \mathcal{E}_{y_1} \). In (2.14), replace \( \{n(j)\} \) by \( \{m(j)\} \) and divide by \( \exp \{ t_{k,m(j)}y_1 \} \). This gives us
\[ \left[ \exp \left\{ t_{k,m(j)}u \right\} - 1 \right] \]
\[ + \sum_{s=1}^{k-1} C_{s,m(j)} \exp \left\{ (t_{s,m(j)} - t_{k,m(j)})y_1 \right\} \left[ \exp \left\{ t_{s,m(j)}u \right\} - 1 \right] \to 0 \]
\[ (u \text{ in } \mathcal{E}_{y_1}). \]

Let \( \mathcal{L} = \mathcal{E}_1 \cdot \mathcal{E}_{y_1} \). We know that \( m(\mathcal{L}) > 0 \). If we restrict \( u \) in (2.15)
and \( y \) in (2.10) to lie in \( \mathcal{L} \), then \( u \) and \( y \) may be identified; so on subtracting (2.15) from (2.10) (with \( n(j) \) replaced by \( m(j) \)) we obtain

\[
(2.16) \sum_{s=1}^{k-1} C_{s,m(j)} \left[ 1 - \exp \left\{ (t_{s,m(j)} - t_{k,m(j)}) y_1 \right\} \right] \left[ \exp \{ t_{s,m(j)} y \} - 1 \right] \rightarrow 0
\]

(\( y \) in \( \mathcal{L} \)).

This relation has only \( k - 1 \) terms, and for it the hypotheses of Lemma 2.2 hold. By our induction assumption, therefore, each coefficient approaches zero. Since (2.12) holds, we must have

\[
\exp \left\{ (t_{1,m(j)} - t_{k,m(j)}) y_1 \right\} \rightarrow 1,
\]

which is contrary to the choice of \( y_1 \).

Thus the induction chain is complete, and the lemma is established.

**Theorem 2.1.** Let \( \{a_{s,n}\}, s=1, \ldots, k, \) be real or complex number sequences, and let the real sequences \( \{r_{s,n}\} \) have the property that none of the sequences \( \{r_{s,n} - r_{p,n}\} (s \neq p) \) has zero as a limit point. If

\[
(2.17) \sum_{s=1}^{k} a_{s,n} \exp \left\{ r_{s,n} x \right\} \rightarrow 0
\]

for all \( x \) on a set \( \mathcal{E} \) of positive measure, then

\[
(2.18) a_{s,n} \rightarrow 0 \quad (s = 1, \ldots, k).
\]

**Remark.** For sequences \( \{r_{s,n}\} \) satisfying the above hypothesis, Theorem 2.1 asserts what may be termed the asymptotic linear independence of the functions \( \exp \left\{ r_{s,n} x \right\}, s=1, \ldots, k \).

If the theorem is false, there is an index \( s \), say \( s=1 \), for which \( a_{1,n} \rightarrow 0 \) is false; so a subsequence \( \{n(j)\} \) of \( \{n\} \) exists, and a positive number \( M \), such that \( \left| a_{1,n(j)} \right| > M \). Replace \( \{n\} \) by \( \{n(j)\} \) in (2.17) and divide by \( a_{1,n(j)} \exp \left\{ r_{1,n(j)} x \right\} \):

\[
(2.19) 1 + \sum_{s=2}^{k} b_{s,n(j)} \exp \left\{ (r_{s,n(j)} - r_{1,n(j)}) x \right\} \rightarrow 0,
\]

where

\[
(2.20) b_{s,n(j)} = \frac{a_{s,n(j)}}{a_{1,n(j)}}.
\]

Take \( x = x_1 \) in (2.19) and subtract from (2.19):

\[
(2.21) \sum_{s=2}^{k} b_{s,n(j)} \exp \{ t_{s,n(j)} x_1 \} \left[ \exp \{ t_{s,n(j)} y \} - 1 \right] \rightarrow 0.
\]
Here
\[ t_{s,n}(j) = r_{s,n}(j) - r_{1,n}(j); \quad y = x - x_1 \quad (x \in \mathcal{E}), \]
so \( y \) ranges over a set \( \mathcal{E}_1 \) of positive measure.

The hypotheses of Lemma 2.2 are satisfied, so
\[ b_{s,n}(j) \to 0 \quad (s = 2, \ldots, k). \]
But this contradicts (2.19). Thus Theorem 2.1 is established.

**COROLLARY 2.1.** Let the sequences \( \{t_{s,n}\} \) satisfy the hypothesis of Lemma 2.2. If a constant \( A \) and constants \( \{a_{s,n}\} \) exist such that
\[
\sum_{s=1}^{k} a_{s,n} \exp \{t_{s,n}x\} \to A
\]
for all \( x \) on a set of positive measure, then
\[
A = 0; \quad a_{s,n} \to 0 \quad (s = 1, \ldots, k).
\]
For, (2.22) can be written
\[
\sum_{s=1}^{k+1} a_{s,n} \exp \{t_{s,n}x\} \to 0,
\]
where
\[
a_{k+1,n} = -A, \quad t_{k+1,n} = 0.
\]
The hypothesis of Theorem 2.1 is fulfilled in (2.24), so \( a_{s,n} \to 0, s = 1, \ldots, k + 1. \)

In Theorem 2.1 the condition on the sequences \( \{r_{s,n}\} \) cannot be weakened. This is shown by the following theorem.

**THEOREM 2.2.** Let the real sequences \( \{r_{s,n}\}, s = 1, \ldots, k, \) be such that at least one of the sequences \( \{r_{s,n} - r_{p,n}\} (s \neq p) \) has zero as a limit point. There exist sequences \( \{a_{s,n}\}, \) at least one of which does not approach zero, such that (2.17) holds for all \( x. \)

We may suppose that \( t_{n(j)} = r_{1,n(j)} - r_{2,n(j)} \to 0 \) as \( j \to \infty. \) Choose \( a_{s,n} = 0, s = 3, \ldots, k, \) and \( a_{1,n} = a_{2,n} = 0 \) for \( n \neq n_1, n_2, \ldots. \) The left side of (2.17) becomes
\[
a_{1,n(j)} \exp \{r_{1,n(j)}x\} + a_{2,n(j)} \exp \{r_{2,n(j)}x\},
\]
and this approaches zero if and only if
\[
a_{1,n(j)} \exp \{t_{n(j)}x\} + a_{2,n(j)} \to 0.
\]
It is clear that if we define \( a_{1,n(j)} = 1, a_{2,n(j)} = -1, \) then (2.27)
does hold for every $x$. Actually, these coefficients can be chosen to be unbounded. For let $R>0$ be given. There exists an $M=M(R)$ such that

$$| \exp \{ t_{n(i)}x \} - 1 | \leq M | t_{n(i)} |$$

for all $|x| \leq R$. On writing the left side of (2.27) as

$$a_{1,n(i)}[\exp \{ t_{n(i)}x \} - 1] + [a_{1,n(i)} + a_{2,n(i)}],$$

we see that (2.27) follows if we choose $a_{1,n(i)}$, $a_{2,n(i)}$ so that

$$(2.28) \quad a_{1,n(i)}t_{n(i)} \to 0, \quad a_{1,n(i)} + a_{2,n(i)} \to 0.$$  

Since $t_{n(i)} \to 0$, conditions (2.28) can be satisfied by sequences $a_{1,n(i)}$, $a_{2,n(i)}$ that are unbounded.

3. Polynomial coefficients. The result of Theorem 2.1 can be extended to the case of polynomial coefficients of bounded degree:

**Theorem 3.1.** Let $\{r_{s,n}\}$, $s=1, \cdots, k$, be real sequences such that none of the sequences $\{r_{s,n}-r_{p,n}\}$ ($s \neq p$) has zero as a limit point. Let $\{P_{s,n}(x)\}$ be real or complex polynomial sequences:

$$(3.1) \quad P_{s,n}(x) = a_{s,0,n} + a_{s,1,n}x + \cdots + a_{s,q_s,n}x^{q_s} \quad (s = 1, \cdots, k)$$

in which $q_s$ is independent of $n$. If

$$(3.2) \quad \sum_{s=1}^{k} P_{s,n}(x) \exp \{ r_{s,n}x \} \to 0$$

for all $x$ on a set $\mathcal{E}$ of positive measure, then

$$(3.3) \quad a_{s,p,n} \to 0 \quad (p = 0, 1, \cdots, q_s; s = 1, \cdots, k).$$

Let

$$(3.4) \quad q = \max \{ q_1, \cdots, q_k \}.$$

If $q=0$ the result follows from Theorem 2.1. Suppose the theorem is false. Then there is an integer $Q>0$ such that whenever $q<Q$ the result is true, but for at least one case with $q=Q$ the theorem is untrue. In each case of failure, with $q=Q$, at least one polynomial coefficient is of degree $Q$. Let $\lambda$ be the number of such polynomials; then there is a positive integer $\Lambda$ with the property that whenever $\lambda<\Lambda$ (and $q=Q$), the theorem is true, but there is a case $\lambda=\Lambda$, $q=Q$ for which it is false.

Let (3.2) be such a case, so that exactly $\Lambda$ polynomials, that we may take to be $P_{s,n}(x), s=1, \cdots, \Lambda$, are of degree $Q$ while all other polynomial coefficients (if any) are of lower degree. Since the theorem is
false for this case, not all the coefficients approach zero as \( n \to \infty \). For each \( n = 1, 2, \cdots \) let
\[
(3.5) \quad \mu_n = \max \{ |a_{\sigma, p, n}| \} \quad (p = 0, 1, \cdots, q; s = 1, \cdots, k).
\]
Then \( \mu_n \) does not approach 0. Therefore there exist values \( s = \sigma, p = p_s \), a positive number \( M \), and a subsequence \( \{n(j)\} \) of \( \{n\} \), such that
\[
(3.6) \quad |a_{\sigma, p, n(j)}| = \mu_n(j) > M \quad (j = 1, 2, \cdots).
\]
Replace \( \{n\} \) by \( \{n(j)\} \) in (3.2) and divide by \( a_{\sigma, p, n(j)} \exp \left\{ r_{1, n(j)} x \right\} \):
\[
(3.7) \quad R_{1, n(j)}(x) + \sum_{s=2}^{k} R_{s, n(j)}(x) \exp \left\{ t_{s, n(j)} x \right\} \to 0 \quad (x \text{ in } \mathcal{E}),
\]
where
\[
t_{s, n(j)} = r_{s, n(j)} - r_{1, n(j)}
\]
and
\[
R_{s, n(j)}(x) = \sum_{p=0}^{q_s} b_{s, p, n(j)} x^p \equiv \frac{1}{a_{\sigma, p, n(j)}} \cdot P_{s, n(j)}(x).
\]
The \( b \)-coefficients are bounded, and \( b_{s, p, n(j)} \equiv 1 \) for all \( j \). Consequently, there exists a subsequence \( \{m(j)\} \) of \( \{n(j)\} \) for which the following limits exist:
\[
(3.8) \quad \lim_{j \to \infty} b_{s, p, m(j)} = b_{s, p} \quad (p = 0, 1, \cdots, q; s = 1, \cdots, k);
\]
and not all of \( b_{s, p} \) are zero, since \( b_{s, q_s} = 1 \).

From (3.7) it follows that
\[
(3.9) \quad R_1(x) + \sum_{s=2}^{k} R_s(x) \exp \left\{ t_{s, m(j)} x \right\} \to 0 \quad (x \text{ in } \mathcal{E}),
\]
where
\[
(3.10) \quad R_s(x) = \sum_{p=0}^{q_s} b_{s, p} x^p \quad (s = 1, \cdots, k).
\]
Moreover, \( R_1(x) \) is of degree \( Q \); for if it is of lower degree, then (3.9) presents a case in which fewer than \( \Lambda \) polynomials are of degree \( Q \), so from the definition of \( \Lambda \) it will follow that the theorem is true for (3.9). Thus all coefficients in all the polynomials approach zero as \( j \to \infty \). But this is contrary to the fact that \( b_{s, q_s} = 1 \). Hence the degree of \( R_1(x) \) must be \( Q \).

We know from the proof of Lemma 2.2 that set \( \mathcal{E} \) contains a point
every neighborhood of which contains a subset of $\mathcal{E}$ of positive measure; and using this fact, we may conclude (as was similarly argued in establishing Lemma 2.2) that there exist distinct numbers $h_1$, $h_2$ such that on defining $\mathcal{E}_1$, $\mathcal{E}_2$ by

$$\mathcal{E}_p = \{y_p = x + h_p, \ x \text{ ranging over } \mathcal{E}\} \quad (p = 1, 2),$$

then $\mathcal{E}_1 \cap \mathcal{E}_2$ is a set of positive measure.

Relation (3.9) may then be written in each of the forms

$$R_1(y_p - h_p)$$

$$+ \sum_{s=2}^{k} R_s(y_p - h_p) \exp \{-t_{s,m(j)} h_p\} \exp \{t_{s,m(j)} y_p\} \to 0$$

$$y_p \in \mathcal{E}_p, \ p = 1, 2).$$

If we consider only points in $\mathcal{E}_s$, then $y_1$ and $y_2$ may be identified:

$$R_1(y - h_p)$$

$$+ \sum_{s=2}^{k} R_s(y - h_p) \exp \{-t_{s,m(j)} h_p\} \exp \{t_{s,m(j)} y\} \to 0$$

$$y \in \mathcal{E}_s, \ p = 1, 2).$$

On subtracting we have

$$[R_1(y - h_1) - R_1(y - h_2)] + \sum_{s=2}^{k} [R_s(y - h_1) \exp \{-t_{s,m(j)} h_1\}$$

$$- R_s(y - h_2) \exp \{-t_{s,m(j)} h_2\} \exp \{t_{s,m(j)} y\} \to 0$$

$$y \in \mathcal{E}_s).$$

Since $R_1(x)$ is of actual degree $Q$, and $Q > 0$, we see that

$$H(y) = [R_1(y - h_1) - R_1(y - h_2)]$$

is a polynomial of degree exactly $Q - 1$, and is therefore not identically zero. But $H(y)$ being of degree less than $Q$, this places (3.13) in the category of cases for which the theorem is true, since now fewer than $\lambda$ polynomials are of degree $Q$. Hence all coefficients approach zero. This is however contrary to the condition that $H(y) \neq 0$.

We have thus arrived at a contradiction, so the assumption that Theorem 3.1 is false is untenable.

4. **Higher dimensions.** We shall now show that the foregoing results extend to the general case of $p$ dimensions. Throughout this section the term *measure* refers to $p$-dimensional measure. Proofs for
the general case usually follow those of the preceding sections, and are accordingly given briefly or not at all.

**Lemma 4.1.** Let \( \{u_{s,n}\}, s = 1, \cdots, p \), be real sequences such that for at least one value of \( s \), \( \{u_{s,n}\} \) does not have zero as its limit. The relation

\[
\lim_{n \to \infty} \exp \left\{ \sum_{s=1}^{p} u_{s,n} x_s \right\} = 1
\]

cannot hold on a set of points \( (x) = (x_1, \cdots, x_p) \) of positive measure.

Assume that the lemma is false, so there is a set \( \mathcal{F} \) of positive measure for which (4.1) holds. Suppose \( s = q \) is the value for which \( u_{q,n} \) does not approach 0. We may then assume that \( \{u_{q,n}\} \) does not have zero as limit point. If a subsequence \( \{n(j)\} \) of \( \{n\} \) exists for which all the sequences \( \{u_{s,n(j)}\}, s = 1, \cdots, p \), are bounded, then there will be a further subsequence \( \{m(j)\} \) for which the following limits exist:

\[
\lim_{j \to \infty} u_{s,m(j)} = l_s \quad (s = 1, \cdots, p),
\]

with \( l_q \neq 0 \). Hence if \( (x) \) is in \( \mathcal{F} \), then \( (x) \) must satisfy one of the equations

\[
\frac{1}{2\pi} \sum_{s=1}^{p} l_s x_s = 0, \pm 1, \pm 2, \cdots.
\]

For each choice of the right side, (4.2) is a hyperplane, and is of measure zero. The totality of planes (4.2) is likewise of zero measure, and so, therefore, is \( \mathcal{F} \), which is contrary to assumption.

There remains to consider the case where for at least one value of \( s \), say \( s = 1 \), and a subsequence \( \{n(j)\} \),

\[
|u_{1,n(j)}| \to \infty.
\]

The remainder of the argument now follows that of Lemma 2.1, with obvious \( p \)-dimensional modifications.

**Lemma 4.2.** Let \( \{t_{s,r,n}\}, s = 1, \cdots, k; \ r = 1, \cdots, p \) be real sequences with the following property: a value \( r = q \) exists such that none of the sequences \( \{t_{s,q,n}\}, \{t_{s,q,n} - t_{s,r,n}\} (s \neq r) \) has zero as a limit point. If real or complex constants \( \{A_{s,n}\} \) exist such that

\[
\sum_{s=1}^{k} A_{s,n} \left[ \exp \left\{ \sum_{r=1}^{p} t_{s,r,n} x_r \right\} - 1 \right] \to 0
\]

for all \( (x) = (x_1, \cdots, x_p) \) on a set \( \mathcal{E} \) of positive measure, then
The proof is like that of Lemma 2.2 with simple modifications that need not be detailed here.

Lemma 4.2 leads directly to the following theorem.

**Theorem 4.1.** Let \( \{a_{s,n}\}, s = 1, \ldots, k, \) be real or complex sequences, and let the real sequences \( \{g_{s,r,n}\}, s = 1, \ldots, k; r = 1, \ldots, p, \) be such that for some value \( r = \omega, \) none of the sequences \( \{q_{s,\omega,n} - q_{s,\omega,n}\} (s \neq \sigma) \) has zero as a limit point. If

\[
(4.4) \quad A_{s,n} \to 0 \quad (s = 1, \ldots, k).
\]

for all \( (x) = (x_1, \ldots, x_p) \) on a set \( \mathcal{E} \) of positive measure, then

\[
(4.5) \quad \sum_{s=1}^{k} a_{s,n} \exp \left\{ \sum_{r=1}^{p} q_{s,r,n}x_r \right\} \to 0
\]

for all \( (x) = (x_1, \ldots, x_p) \) on a set \( \mathcal{E} \) of positive measure, then

\[
(4.6) \quad a_{s,n} \to 0 \quad (s = 1, \ldots, k).
\]

The proof follows an earlier one (Theorem 2.1), as does the next result:

**Corollary 4.1.** Let the sequences \( \{t_{s,r,n}\} \) satisfy the hypothesis of Lemma 4.2. If a constant \( A \) and constants \( \{a_{s,n}\} \) exist such that

\[
(4.7) \quad \sum_{s=1}^{k} a_{s,n} \exp \left\{ \sum_{r=1}^{p} t_{s,r,n}x_r \right\} \to A
\]

for all \( (x) \) on a set of positive measure, then

\[
(4.8) \quad A = 0; \quad a_{s,n} \to 0 \quad (s = 1, \ldots, k).
\]

Finally, we have

**Theorem 4.2.** Let \( \{g_{s,r,n}\}, s = 1, \ldots, k; r = 1, \ldots, p, \) be real sequences satisfying the hypothesis of Theorem 4.1. Let \( \{P_{s,n}(x_1, \ldots, x_p)\} \) be real or complex polynomial sequences:

\[
(4.9) \quad P_{s,n}(x_1, \ldots, x_p) = \sum_{h_1+\cdots+h_p=0}^{e_s} a_{s,n;h_1,\ldots,h_p} x_1^{h_1} \cdots x_p^{h_p} \quad (s = 1, \ldots, k),
\]

in which \( e_s \) is independent of \( n. \) If

\[
(4.10) \quad \sum_{s=1}^{k} P_{s,n}(x_1, \ldots, x_p) \exp \left\{ \sum_{r=1}^{p} q_{s,r,n}x_r \right\} \to 0
\]

for all \( (x) \) on a set \( \mathcal{E} \) of positive measure, then
\[ a_{s,n;h_1,\ldots,h_p} \to 0 \quad (0 \leq h_1 + \cdots + h_p \leq \epsilon_s; s = 1, \ldots, k). \]

Up to a point the proof is patterned after that of Theorem 3.1. When the equivalent of (3.13) is obtained, however, we can no longer assert that \( H(y_1, \ldots, y_p) \) is not identically zero simply from the fact that two distinct sets \( (h)_1 = (h_{11}, \ldots, h_{1p}), (h)_2 = (h_{21}, \ldots, h_{2p}) \) exist such that

\[
(4.12) \quad H(y_1, \ldots, y_p) = R_1(y_1 - h_{11}, \ldots, y_p - h_{1p})
- R_1(y_1 - h_{21}, \ldots, y_p - h_{2p}).
\]

In fact, nonconstant polynomials in more than one variable exist that are "periodic." We avoid this difficulty by observing that for a fixed point \( (h)_1 \), the point \( (h)_2 \) can be chosen arbitrarily on a set of positive measure. Examination of the proof of Lemma 2.2 shows this. Now if a polynomial \( L(x_1, \ldots, x_p) \) has the property that

\[
L(x_1 + c_1, \ldots, x_p + c_p) = L(x_1, \ldots, x_p)
\]

for all sets \( (c) = (c_1, \ldots, c_p) \) on a set of positive measure, then surely \( L = \text{constant} \).

In our case, therefore, if \( H(y_1, \ldots, y_p) \equiv 0 \) for all possible choices of \( (h)_2 \), then \( R_1 \) is a constant, contrary to the fact that its degree is \( Q > 0 \) (cf. Theorem 3.1). The remainder of the proof offers no difficulty.

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