NICHOLSON’S INTEGRAL FOR $J_n^2(z) + Y_n^2(z)$

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The integral in question is

(1) \[ J_n^2(z) + Y_n^2(z) = \frac{8}{\pi^2} \int_0^\infty K_0(2z \sinh t) \cosh 2nt \, dt, \]

and its validity for arbitrary complex $n$ when the real part of $z$ is positive is proved in [1, pp. 441-444] with the help of Hardy’s theory of generalized integrals and integrations over contours in the complex plane. It is the purpose of this paper to give a much more elementary proof of (1).

We begin by observing [1, p. 146] that if $D = z(d/dz)$, then three linearly independent solutions of the equation

(2) \[ [D(D^2 - 4n^2) + 4z^2(D + 1)]y = 0 \]

are $J_n^2(z)$, $Y_n^2(z)$ and $J_n(z)Y_n(z)$. Equation (2) may be written as

(3) \[ z^2y''' + 3zy'' + (1 - 4n^2 + 4z^2)y' + 4zy = 0. \]

We shall now show that $y(z) = \int_0^\infty K_0(2z \sinh t) \cosh 2nt \, dt$ is a solution of (3). When the real part of $z$ is positive it is clear that $K_0(2z \sinh t)$ is sufficiently small at $\infty$ to permit us to differentiate under the integral sign as many times as we please. Therefore,

(4) \[ y'(z) = 2 \int_0^\infty \sinh tK_0'(2z \sinh t) \cosh 2nt \, dt. \]

If we make use of the differential equation

(5) \[ xK_0'(x) + K_0(x) - xK_0(x) = 0 \]

satisfied by $K_0(x)$, then we find that

\[ \gamma'' = \int_0^\infty \{ 4 \sinh^2 tK_0(2z \sinh t) - 2z^{-1} \sinh tK_0'(2z \sinh t) \} \cosh 2nt \, dt, \]

\[ \gamma''' = \int_0^\infty \{ (8 \sinh^3 t + 4z^{-2} \sinh t)K_0'(2z \sinh t) \]

\[ - 4z^{-1} \sinh^2 tK_0(2z \sinh t) \} \cosh 2nt \, dt. \]

It follows that

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1 Numbers in brackets refer to the reference cited at the end of the paper.

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\[ z^2y''' + 3zy'' + (1 + 4z^2)y' + 4zy \]

(6) \[ = \int_0^\infty \left\{ 4z^2 \sinh 2t \cosh tK_0'(2z \sinh t) \right. \]

\[ + 4z \cosh 2tK_0(2z \sinh t) \} \cosh 2ntdt. \]

If now (4) is integrated by parts and use is made of (5) we find that

\[ 4n^2y' = -4nz \int_0^\infty \sinh 2tK_0(2z \sinh t) \sinh 2ntdt, \]

whence another integration by parts shows that \[ 4n^2y' \] is equal to the right-hand side of (6). Therefore \[ y(z) \] is a solution of (3). Consequently, there exist constants \[ A, B, C \] such that

(7) \[ y(z) = Az^2 + BY^2_n + CJ_nY_n(z). \]

We shall now show that

(8) \[ \lim_{z \to \infty} zy(z) = \lim_{z \to \infty} \int_0^\infty zK_0(2z \sinh t) \cosh t dt = \frac{\pi}{4}, \]

the last equality being a consequence of the result \( [1, \text{p. 388}] \)

\[ \int_0^\infty K_0(u) du = \frac{\pi}{2}. \]

In (8), \( z \) is restricted to real values. In fact, the difference of the integrands in the limitands in (8) is

\[ F(z, t) = zK_0(2z \sinh t)(\cosh 2nt - \cosh t). \]

Now \( x^{1/2}e^xK_0(x) \) is bounded on \( (0, \infty) \), so that

\[ |F(z, t)| \leq A_0(z \text{csch} t)^{1/2}e^{-2zs \sinh t} | \cosh 2nt - \cosh t|. \]

Moreover, \( \text{csch} t \leq 1/t \) and the mean value theorem shows that

\[ | \cosh 2nt - \cosh t| \leq (2|n| + 1)t(\sinh 2|n| t + \sinh t), \]

whence we see that

\[ |F(z, t)| \leq A_1(zt)^{1/2}e^{-2z \sinh t}(\sinh 2|n| t + \sinh t). \]

We can suppose that \( z \geq 1 \). Since \( \sinh t \geq t \) and \( (zt)^{1/2}e^{-zt} \) is bounded, we find that

\[ |F(z, t)| \leq A_2(\sinh 2|n| t + \sinh t)e^{-z \sinh t} \]

\[ \leq A_3(\sinh 2|n| t + \sinh t)e^{-\sinh t}. \]
Therefore, \( F(z, t) \) converges dominatedly to zero as \( z \) approaches \( \infty \), and this suffices to prove (8).

It is known [1, p. 199] that
\[
J_n(z) = (2/\pi z)^{1/2} \cos \left( z - \frac{n\pi}{2} - \frac{\pi}{4} \right) + O(z^{-3/2}),
\]
\[
Y_n(z) = (2/\pi z)^{1/2} \sin \left( z - \frac{n\pi}{2} - \frac{\pi}{4} \right) + O(z^{-3/2}).
\]

From (7) we conclude that
\[
\frac{\pi z y(z)}{2} = A + (B - A) \sin^2 \left( z - \frac{n\pi}{2} - \frac{\pi}{4} \right)
\]
\[
+ \frac{C}{2} \sin \left( 2z - n\pi - \frac{\pi}{2} \right) + O(z^{-1}).
\]

This result is incompatible with (8) unless \( A = \pi^2/8 \), \( B = A \), \( C = 0 \), and in this case \( y(z) = (\pi^2/8) \{ J_0^2(z) + Y_0^2(z) \} \). This completes the proof of (1).

Reference


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