ON THE SUM OF CUBES

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Large capital letters \( A, B, \cdots \) (without or with subscripts) will represent integers of the quadratic number field \( Ra(\rho) \) where \( \rho = \left(-1 + (−3)^{1/2}\right)/2 \). Small latin letters \( a, b, \cdots \) represent rational integers, and the conjugate of a number \( X \) is denoted by \( \bar{X} \).

The object of this paper is to give a method for obtaining the complete rational integer solution for the diophantine equations of the form

\[
\sum_{i=1}^{m} x_i^3 = 0, \quad m > 3.
\]

This equation with \( m \) even, \( m = 2n \), can be written as \( \sum_{i=1}^{2n}(X_i + \bar{X}_i)X_i \bar{X}_i = 0 \) where

\[
X_i = z_{2i-1} + \rho(z_{2i-1} - z_{2i})
\]

and thus the problem of solving (1) in this case is reduced to that of finding all the integers \( x_i, X_i \) satisfying the equations

(3)
\[
\sum_{i=1}^{n} x_i X_i \bar{X}_i = 0,
\]

(4)
\[
x_i = X_i + \bar{X}_i \quad (i = 1, 2, \cdots, n)
\]

and (2). When \( m \) is odd, \( m = 2n - 1 \), we solve the system \((a)\) consisting of (3), \( x_n = X_n = z_{2n-1} \) and (2), (4) for \( i = 1, 2, \cdots, n-1 \).

The resolution of these two systems hinges on techniques developed by E. T. Bell \([2]\), being equivalent to the resolution of a system of multiplicative equations and a system of linear homogeneous equations in \( Ra \) in which the number of unknowns always exceeds the number of equations.

In solving (1) the following equations appear:

(5)
\[
x_1 y_1 + x_2 y_2 + \cdots + x_n y_n = 0
\]

in which the \( x_i, y_i \) \((i = 1, 2, \cdots, n)\) are \( 2n \) independent variables;

(6)
\[
a_{i1} x_1 + \cdots + a_{in} x_n = 0 \quad (i = 1, 2, \cdots, m \leq n - 1)
\]

in which the \( n \) independent variables \( x_i \) are to be solved in terms of the coefficients \( a_{ij} \);

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\[ A_1B_1C_1 = \cdots = A_{n-1}B_{n-1}C_{n-1} \]

in \(3(n-1)\) independent variables \(A_i, B_i, C_i\);

\[ p_1A_1A_1 = p_2A_2A_2 = \cdots = p_nA_nA_n \]

in which the \(p_i, A_i\) are independent variables.

The solution [5, p. 20 (13)] of (5) is

\[ x_i = a_i, \quad y_i = -\sum_{j=1}^{i-1} a_i b_{i,j} + \sum_{j=i-1}^{n-1} a_{i+j} b_{i,j+1} \]

for \(i = 1, 2, \ldots, n\) with the convention that a sum in which the lower limit exceeds the upper is vacuous. Note that there are \((n^2-n)/2\) free parameters \(b_{i,k}\) and \(n+1\) free parameters \(a, a_i\).

If the system (6) is of rank \(m\) then its complete solution [3] in determinantal form is written down as follows. Let \(e_j\) be the determinant obtained by deleting the \(j\)th column from the matrix of the coefficients of the system consisting of (6) and the equations

\[ c_{i1}x_1 + c_{i2}x_2 + \cdots + c_{in}x_n = 0 \quad (i = 1, 2, \ldots, n - m - 1) \]

in which the \(c_{ij}\) are arbitrary rational integers. Then

\[ x_j = (-)^j t e_j / e \quad (j = 1, 2, \ldots, n) \]

where \(t\) is an arbitrary integer and \(e = (e_1, e_2, \ldots, e_n)\).

The system (7) is recursive and can be solved completely by the algorithm of reciprocal arrays [1] since the integers of \(Ra(\rho)\) form a principal ideal ring. System (7) is equivalent to the equations

\[ A_iB_iC_i = A_{n-1}B_{n-1}C_{n-1} \quad (i = 1, 2, \ldots, n - 2) \]

The solution of the typical equation is

\[ A_i = A_{i1}B_{i1}C_{i1}, \quad A_{n-1} = A_{i1}H_{i1}F_{i1}, \]
\[ B_i = D_{i1}E_{i1}F_{i1}, \quad B_{n-1} = D_{i1}B_{i1}J_{i1}, \]
\[ C_i = G_{i1}H_{i1}J_{i1}, \quad C_{n-1} = G_{i1}E_{i1}C_{i1}. \]

Then the values of \(A_{n-1}\) are equated, also those of \(B_{n-1}\), and those of \(C_{n-1}\). The resulting three systems are each of the type (7) with \(n-2\) in place of \(n-1\). By repetitions of the process the solution of (7) involving \((n-1) = 3^{n-1}\) free parameters \(K_1, K_2, \ldots, K_{(n-1)}\) is obtained in the form

\[ A_i = K_1\Psi_i(K_4, \ldots, K_{(n-1)}), \quad B_i = K_2\Psi_i(K_4, \ldots, K_{(n-1)}), \]
\[ C_i = K_3\Theta_i(K_4, \ldots, K_{(n-1)}) \]
where each of $\Phi_i, \Psi_i, \Theta_i$ is a monomial in $3^{n-2}-1$ of the parameters $K_3, K_4, \ldots, K_{(n-1)}$ each occurring only once in a particular $\Phi_i, \Psi_i, \Theta_i$ and

$$K_1K_2K_3\Phi_i(K_4, \ldots, K_{(n-1)})\Psi_i(K_4, \ldots, K_{(n-1)})\Theta_i(K_4, \ldots, K_{(n-1)}) = K_1K_2 \cdots K_{(n-1)}$$

for $i = 1, 2, \ldots, n-1$.

The resolution of (8) is also recursive. This system is equivalent to the $n-1$ equations $p_nA_n\bar{A}_n = p_iA_i\bar{A}_i$ ($i = 1, 2, \ldots, n-1$). The solution [4, Theorem 1] of the typical equation is

$$p_i = \tilde{t}_i \tilde{V}_i \tilde{V}_i, \quad p_n = \tilde{t}_i \tilde{L}_i \tilde{L}_i,$$

$$A_i = S_i \tilde{U}_i L_i \tilde{L}_i, \quad A_n = S_i \tilde{U}_i V_i \tilde{V}_i.$$  

Then the values of $p_n$ are equated and also those of $A_n$ which yield the two independent systems

\begin{align*}
(10) & \quad \tilde{t}_i \tilde{L}_i \tilde{L}_i = \tilde{t}_{n-1,1} L_{n-1,1} \tilde{L}_{n-1,1} \\
(11) & \quad S_i \tilde{U}_i V_i \tilde{V}_i = S_{n-1,1} U_{n-1,1} V_{n-1,1}
\end{align*}

for $i = 1, 2, \ldots, n-2$.

System (10) is of the type (8) with $n-1$ in place of $n$; system (11) is of the type (7) and its solution is therefore

$$S_i = K_1\Phi_i(K_4, \ldots, K_{(n-1)}), \quad U_i = K_2\Psi_i(K_4, \ldots, K_{(n-1)}),$$

$$V_i = K_3\Theta_i(K_4, \ldots, K_{(n-1)}),$$

for $i = 1, 2, \ldots, n-1$.

Hence all integral solutions of (8) are given by

$$A_i = K_1\Phi_i K_2\Psi_i L_i \tilde{L}_i, \quad A_n = K_1\Phi_i K_2\Psi_i K_3\Theta_i,$$

$$p_i = t_i K_3 K_4 \Theta_i \tilde{L}_i, \quad p_n = t_i K_2 K_4 \Theta_i \tilde{L}_i,$$

with the condition (10).

The process just applied to (8) is now repeated on (10) which will yield parametric expressions for $t_i, L_i \tilde{L}_i$ similar to those for $p_i, A_i$ respectively subject to systems of the type (10) and (11) with $n-1$ replaced by $n-2$. We note that this process must finally yield the solutions in the form

$$A_i = KE_i, \quad p_i = dM_i \bar{M}_i,$$

where the $E_i$ and also the $M_i$ are products of integers of $Ra(p)$; all the $E_i$, and also all the $M_i$ are of course not independent but they
are each independent of $K$.

The resolution of (3) now follows. Applying (5) to (3) yields

$$X_iX_i = a_ia_i, \quad x_i = -\sum_{j=1}^{i-1} a_j b_{j,i} + \sum_{j=1}^{n-i} a_{i+j} b_{i,i+j}$$

for $i = 1, 2, \ldots, n$.

Hence we must now solve the system of equations

$$X_iX_i = a_ia_i \quad (i = 1, 2, \ldots, n).$$

The solution of the typical equation is

$$X_i = \rho_i A_i B_i, \quad a = \rho_i A_i \bar{A}_i, \quad a_i = \rho_i B_i \bar{B}_i$$

in free parameters $\rho_i, A_i, B_i$.

Equating the value of $a$ yields the system (8) and hence by (12) $A_i = kE_i, \quad \rho_i = dM_i \bar{M}_i$ and therefore the complete solution of (3) is given by

$$X_i = K R_i, \quad x_i = -\sum_{j=1}^{i-1} a_j b_{j,i} + \sum_{j=1}^{n-i} a_{i+j} b_{i,i+j}$$

for $i = 1, 2, \ldots, n$ where

$$R_i = dM_i \bar{M}_i B_i E_i, \quad a_i = dM_i \bar{M}_i B_i \bar{B}_i.$$

The resolution of (1) now follows. Put $K = k_1 + \rho k_2, \quad R_i = r_i + \rho s_i$; then $KR_i + \bar{K} \bar{R}_i = k_1 (2r_i - s_i) - k_2 (r_i + s_i)$. Then all the $X_i, x_i$ satisfying (3) and (4) simultaneously are given by (13) where values are assigned to the parameters which determine $R_i, a_i$ in (14) and then the $(n^2 - n + 4)/2$ unknowns $k_1, k_2, b_{ij}$ are determined from the $n$ linear homogeneous equations

$$k_1 (2r_i - s_i) - k_2 (r_i + s_i) + \sum_{j=1}^{i-1} a_j b_{j,i} - \sum_{j=1}^{n-i} a_{i+j} b_{i,i+j} = 0$$

for $i = 1, 2, \ldots, n$, a system of the type (6).

Substitute this value of $X_i$ in (2). Equate real and imaginary parts and all the rational integer solutions of (1) with $m = 2n$ will be obtained.

To solve (1) when $m$ is odd, $m = 2n - 1$, we proceed much as above, replacing system (15) by system ($\alpha$) which is equivalent to (13) and the equations

$$k_1 (2r_i - s_i) - k_2 (r_i + s_i) + \sum_{j=1}^{i-1} a_j b_{j,i} + \sum_{j=1}^{m-i} a_{i+j} b_{i,i+j} = 0$$
for \( i = 1, 2, \ldots, n - 1, \)

\[
k_1 r_n - k_2 s_n + \sum_{j=1}^{n-1} a_j b_{j,n} = 0, \quad k_1 s_n + k_2 (r_n - s_n) = 0,
\]

a linear homogeneous system of \( n+1 \) equations in \((n^2 - n + 4)/2\) unknowns \( k_1, k_2, b_{ij} \). The corresponding \( X_i \) given by (13) are substituted in (2) for \( i = 1, 2, \ldots, n - 1 \) and we put \( X_n = z_{2n-1} \).

We conclude by exhibiting the complete solution of \( \sum_{i=1}^{3} z_i^2 = 0 \) in terms of integers of \( Ra(\rho) \).

The complete solution of \( p_1 A_1 A_1 = p_2 A_2 A_2 = p_3 A_3 A_3 \) is given by (12) where

\[
M_1 = GHJT, \quad M_2 = GFLN, \quad M_3 = LPST,
E_1 = CDPLNPS, \quad E_2 = CDHPQST, \quad E_3 = CDFGHJNQ,
\]

and all the parameters are arbitrary. Hence from (14) we get the corresponding values of \( a_i, R_i = r_i + ps_i \), where \( d, B_i \) are arbitrary. In this case (15) is a linear homogeneous system of 3 equations in 5 unknowns. To complete this linear system for resolution we adjoin the single equation

\[
m_1 b_{23} + m_2 b_{13} + m_3 b_{12} + m_4 k_1 + m_5 k_2 = 0
\]

with arbitrary coefficients \( m_i \).

Hence with \( a_i, r_i, s_i \) as found above, (9) gives

\[
e_k_1 = t(a_1 m_1 - a_2 m_2 + a_3 m_3)(a_1(r_1 + s_1) + a_2(r_2 + s_2) + a_3(r_3 + s_3)),
\]

\[
e_k_2 = t(a_1 m_1 - a_2 m_2 + a_3 m_3)(a_1(2r_1 - s_1) + a_2(2r_2 - s_2) + a_3(2r_3 - s_3)).
\]

Then from (2) and (13) for \( i = 1, 2, 3 \)

\[
z_{2i-1} = k_1 r_i - s_i k_2, \quad z_{2i} = k_1 (r_i - s_i) - k_2 r_i.
\]

REFERENCES


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