RECURSIVE PROPERTIES OF TRANSFORMATION GROUPS. II

W. H. GOTTSCALK

The purpose of this note is to sharpen a previous result on the transmission of recursive properties of a transformation group to certain of its subgroups. [See Recursive properties of transformation groups, by W. H. Gottschalk and G. A. Hedlund, Bull. Amer. Math. Soc. vol. 52 (1946) pp. 637–641.]

Let $T$ be a multiplicative topological group with identity $e$. A subset $R$ of $T$ is said to be relatively dense provided that $T = RK$ for some compact set $K$ in $T$.

**Lemma 1.** If $R$ is a relatively dense closed semi-group $(RR \subseteq R)$ in $T$, then $R$ is a subgroup of $T$.

**Proof.** Suppose $r \in R$ and $U$ is a neighborhood of $e$. It is sufficient to show that $r^{-1}U \cap R \neq \emptyset$. Let $V$ be a neighborhood of $e$ for which $VV^{-1} \subseteq U$ and let $K$ be a compact set in $T$ for which $T = RK$. There exists a finite collection $F$ of right translates of $V$ which covers $K$. Choose $k_0 \in K$. Now $r^{-1}k_0 = r_1k_1$ for some $r_1 \in R$ and some $k_1 \in K$. Again $r^{-1}k_1 = r_2k_2$ for some $r_2 \in R$ and some $k_2 \in K$. This may be continued. Thus there exist sequences $k_0, k_1, \ldots$ in $K$ and $r_1, r_2, \ldots$ in $R$ such that $r^{-1}k_i = r_{i+1}k_{i+1}$ ($i = 0, 1, \ldots$). Select integers $m$ and $n$ ($0 \leq m < n$) and an element $V_0$ of $F$ such that $k_m, k_n \in V_0$. Now $r^{-1}k_mk_n^{-1} = (r^{-1}k_mk_{m+1}^{-1}) (k_{m+1}k_{m+2}^{-1}) \cdots (k_{n-1}k_n^{-1}) = r_{m+1}r_{m+2} \cdots r_r \in R$. Also $r^{-1}k_mk_n^{-1} \in r^{-1}V_0V_0^{-1} \subseteq r^{-1}VV^{-1} \subseteq r^{-1}U$. Hence $r^{-1}U \cap R \neq \emptyset$ and the proof is completed.

Now let $T$ act as a transformation group on a topological space $X$. That is to say, suppose that to $x \in X$ and $t \in T$ is assigned a point, denoted $xt$, of $X$ such that: (1) $xe = x$ ($x \in X$); (2) $(xt)s = x(ts)$ ($x \in X, t, s \in T$); (3) The function $xt$ defines a continuous transformation of $X \times T$ into $X$. We assume for the remainder of the paper that $x$ is a fixed point of $X$, $T$ is locally compact and $S$ is a relatively dense invariant subgroup of $T$. Let $\Sigma$ denote the maximal subset of $T$ for which $x\Sigma \subseteq (xS)^*$. The function $xt$ defines a continuous transformation of $X \times T$ into $X$. We assume for the remainder of the paper that $x$ is a fixed point of $X$, $T$ is locally compact and $S$ is a relatively dense invariant subgroup of $T$. Let $\Sigma$ denote the maximal subset of $T$ for which $x\Sigma \subseteq (xS)^*$ where the star denotes the closure operator.

**Lemma 2.** The set $\Sigma$ is a closed subgroup of $T$ which contains $S$.

**Proof.** Obviously $\Sigma \subseteq S$. From $x\Sigma^* \subseteq (x\Sigma)^* \subseteq (xS)^*$ we conclude that $\Sigma$ is closed. By Lemma 1 it is now enough to show that $\Sigma$ is a

---

Presented to the Society, September 2, 1947; received by the editors June 30, 1947.

381
semi-group. Suppose \( \sigma, \tau \in \Sigma \). From \( x\sigma \in (xS)^* \) it follows that \( x\sigma \tau \in (xS)^* \tau \subset (xS^r)^* \). From \( x\tau S \in (xS)^* \) it follows that \( x\tau S \subset (xS)^* S \subset (xSS)^* \subset (xS)^* \). Hence \( x\sigma \in (xS)^* \). Thus \( \sigma \tau \in \Sigma \) and the proof is completed.

**Lemma 3.** If \( W \) is a neighborhood of \( e \), then \( x \in (x[T - \Sigma W])^* \).

**Proof.** We first show that if \( t \in T - \Sigma \), then \( x \in (x\Sigma V_0)^* \) for some neighborhood \( V_0 \) of \( t \). Suppose \( t \in T - \Sigma \). Since \( t^{-1} \in \Sigma \) by Lemma 2, \( x t^{-1} \in (x \Sigma)^* \) and \( x \in (x \Sigma t)^* \). There are neighborhoods \( U \) of \( x \) and \( V \) of \( e \) such that \( V = V^{-1} \) and \( UV \cap x\Sigma t V = \emptyset \). It follows that \( U \cap x\Sigma t V = \emptyset \). Define \( V_0 = tV \).

We may assume \( W \) is open. Define \( N = K - \Sigma W \) where \( K \) is a compact set in \( T \) such that \( T = SK \). Using Lemma 2 we conclude that \( T = SK \subset S(N \cup \Sigma W) \subset S \cap S \cap W \subset \Sigma N \cup \Sigma W \). Hence \( T - \Sigma W = \Sigma N \). By the preceding paragraph, to each \( n \in N \) there corresponds a neighborhood \( V_n \) of \( n \) such that \( x \in (x\Sigma V_n)^* \). Since finitely many of the \( V_n \) cover \( N \), \( x \in (x \Sigma N)^* \). The proof is completed.

**Lemma 4.** If \( U \) is a neighborhood of \( x \), then there exists a compact set \( M \) in \( T \) such that \( xM \subset U \) and \( \Sigma \subset SM^{-1} \).

**Proof.** Define \( N = K \cap \Sigma \) where \( K \) is a compact set in \( T \) such that \( T = SK \). If \( n \in N \), then \( xn \in (xS)^* \) and \( x \in (xSn^{-1})^* \). Thus \( n \in N \) implies the existence of \( s_n \in S \) such that \( xsn^{-1} \in \text{init} U \) and hence the existence of a compact neighborhood \( W_n \) of \( s_nn^{-1} \) such that \( xW_n \subset U \).

Since \( N \) is compact by Lemma 2, there is a finite subset \( F \) of \( N \) for which \( N \subset \bigcup_{n \in F} W_n^{-1}s_n \). Define \( M = \bigcup_{n \in F} W_n \). Clearly \( xM \subset U \). Using Lemma 2 we conclude that \( \Sigma \subset SN \subset SM^{-1} \). The proof is completed.

Let there be distinguished in \( T \) certain sets, called *admissible*, which satisfy this condition: If \( A \) is an admissible set and if \( B \) is a set in \( T \) such that \( A \subset BK \) for some compact set \( K \) in \( T \), then \( B \) is an admissible set. A subgroup \( R \) of \( T \) is said to be *recursive* at \( x \) provided that to each neighborhood \( U \) of \( x \) there corresponds an admissible set \( A \) such that \( A \subset R \) and \( xA \subset U \).

**Lemma 5.** If \( T \) is recursive at \( x \), then \( \Sigma \) is recursive at \( x \).

**Proof.** Let \( U \) be a neighborhood of \( x \). There are neighborhoods \( V \) of \( x \) and \( W \) of \( e \) such that \( W = W^{-1} \), \( W \) is compact and \( VW \subset U \).

By Lemma 3 we may suppose that \( V \cap x(T - \Sigma W) = \emptyset \). There exists an admissible set \( A \) in \( T \) such that \( xA \subset V \). Clearly \( A \subset \Sigma W \) and \( xA W \subset U \). Define \( B = \Sigma \cap A W \). Since \( A \subset BW \), \( B \) is an admissible set. Also \( B \subset \Sigma \) and \( xB \subset U \). The proof is completed.
**Lemma 6.** If $\Sigma$ is recursive at $x$, then $S$ is recursive at $x$.

**Proof.** Let $U$ be an open neighborhood of $x$. By Lemma 4 there exists a compact set $M$ in $T$ such that $xM \subseteq U$ and $\Sigma \subseteq SM^{-1}$. Let $V$ be a neighborhood of $x$ for which $VM \subseteq U$. There exists an admissible set $A$ such that $A \subseteq \Sigma$ and $xA \subseteq V$. Hence $xAM \subseteq U$. Define $B = S \cap AM$. Since $A \subseteq BM^{-1}$, $B$ is an admissible set. Also $B \subseteq S$ and $xB \subseteq U$. The proof is completed.

The following theorem is an immediate consequence of Lemmas 5 and 6.

**Theorem.** If $T$ is recursive at $x$, then $S$ is recursive at $x$.

An interpretation of admissibility arises if we define an admissible subset of $T$ to be a relatively dense subset of $T$. The term "recursive" is then replaced by "almost periodic." For other applications, see the paper cited above.

University of Pennsylvania

**FIXED POINT THEOREMS FOR INTERIOR TRANSFORMATIONS**

O. H. HAMILTON

If $M$ is a bounded continuum in a Euclidean plane $E$ which does not separate $E$ and $T$ is an interior continuous transformation of $M$ onto a subset of $E$ which contains $M$, does $T$ leave a point of $M$ invariant? It is the purpose of this paper to answer this question in the affirmative for certain types of locally connected continua.

Using a notation introduced by Eilenberg [2, p. 168] a continuum $M$ will be said to have property (b) provided every continuous transformation of $M$ into the unit circle $S$ in the Cartesian plane, with center at $o$, is homotopic to a constant mapping, that is, a transformation which transforms each point of $M$ into a single point of $S$. If $T$ is a continuous transformation of a subset $A$ of the plane $E$ into a subset $B$ of $E$, then for each point $x$ of $A$ let $T'(x)$ be the point $y$ of $S$ such that the directed line segment $oy$ is parallel in direction and sense to the directed line segment $x, T(x)$. Then $T'$ will be referred to as the transformation of $A$ into $S$ derived from $T$. Such a transformation

---

Presented to the Society, September 3, 1947; received by the editors June 10, 1947.

1 Numbers in brackets refer to the references cited at the end of the paper.