ON THE DISTRIBUTION OF THE MAXIMUM OF SUCCESSIVE CUMULATIVE SUMS OF INDEPENDENTLY BUT NOT IDENTICALLY DISTRIBUTED CHANCE VARIABLES

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1. Introduction. Let $X_1, X_2, \cdots,$ and so on be a sequence of chance variables and let $S_i$ denote the sum of the first $i$ $X$'s, that is,

(1.1) \[ S_i = X_1 + \cdots + X_i \quad (i = 1, 2, \cdots, \text{ad inf}). \]

Let $M_N$ denote the maximum of the first $N$ cumulative sums $S_1, \cdots, S_N$, that is,

(1.2) \[ M_N = \max (S_1, \cdots, S_N). \]

The distribution of $M_N$, in particular the limiting distribution of a suitably normalized form of $M_N$, has been studied by Erdös and Kac [1] and by the author [2] in the special case when the $X$'s are independently distributed with identical distributions.

In this note we shall be concerned with the distribution of $M_N$ when the $X$'s are independent but not necessarily identically distributed. In particular, the mean and variance of $X_i$ may be any functions of $i$.

In §2 lower and upper limits for $M_N$ are obtained which yield particularly simple limits for the distribution of $M_N$ when the $X$'s are symmetrically distributed around zero.

In §3 the special case is considered when $X_i$ can take only the values 1 and $-1$ but the probability $p_i$ that $X_i = 1$ may be any function of $i$. The exact probability distribution of $M_N$ for this case is derived and expressed as the first row of a product of $N$ matrices.

The limiting distribution of $M_N/N^{1/2}$ is treated in §4. Since the interesting limiting case arises when the mean of $X_i$ ($i \leq N$) is not only a function of $i$ but also a function of $N$, we have to introduce a double sequence of chance variables. That is, for any $N$ we consider a sequence of $N$ chance variables $X_{N1}, \cdots, X_{NN}$. Let $\mu_{Ni}$ denote the mean and $\sigma_{Ni}$ the standard deviation of $X_{Ni}$. Let, furthermore, $S_{Ni}$ denote the sum $X_{N1} + \cdots + X_{Ni}$ and $M_N$ the maximum of $S_{N1}, \cdots, S_{NN}$. With the help of a method used by Erdös and Kac [1], the following theorem is established in §4:

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1 Numbers in brackets refer to the references cited at the end of the paper.

422
Theorem 1.1 Let \( \{X_{Ni}\} \) and \( \{X_{Ni}^*\} \) \( (i = 1, \cdots, N; N = 1, 2, \cdots, \) ad inf.\) be two sequences of chance variables such that the following conditions are fulfilled:

(a) The \( X \)'s are independently distributed.
(b) The sequence \( \{\sigma_{Ni}\} \) \( (i = 1, \cdots, N; N = 1, 2, \cdots, \) ad inf.\) has a positive lower bound and a finite upper bound.
(c) \( \mu_{Ni}^*N^{1/2} \) is a bounded function of \( i \) and \( N \).
(d) The third absolute moment of \( X_{Ni} \) is a bounded function of \( i \) and \( N \).
(e) The conditions (a)–(d) remain valid if we replace \( X_{Ni} \) by \( X_{Ni}^* \).
(f) The equation

\[
\lim_{N \to \infty} \left[ \frac{\mu_{Ni}^* + \cdots + \mu_{N,i}}{\sigma_{Ni}^2 + \cdots + \sigma_{N,i}^2} \right] = 0
\]

holds for all \( i \) and \( N \) where \( \mu_{Ni}^* \) is the mean and \( \sigma_{Ni}^* \) is the standard deviation \( X_{Ni}^* \) and \( j_i \) is the smallest positive integer for which

\[
\frac{\sigma_{Ni}^2 + \cdots + \sigma_{N,i}^2}{\sigma_{Ni}^2 + \cdots + \sigma_{NN}^2} \leq \frac{\sigma_{Ni}^2 + \cdots + \sigma_{N,i}^2}{\sigma_{Ni}^2 + \cdots + \sigma_{NN}^2}
\]

Let

\[
M_N = M_N^* \left( \frac{\sigma_{N1}^2 + \cdots + \sigma_{NN}^2}{\sigma_{N1}^2 + \cdots + \sigma_{NN}^2} \right)^{1/2}
\]

where \( M_N^* \) is the same function of the \( X^* \)'s as \( M_N \) is of the \( X \)'s. Then for any positive \( \epsilon \) we have

(1.5) \( \lim \inf_{N \to \infty} \left[ \text{prob} \left\{ M_N < cN^{1/2} \right\} - \text{prob} \left\{ M_N < (c - \epsilon)N^{1/2} \right\} \right] \geq 0 \)
and

(1.6) \( \lim \inf_{N \to \infty} \left[ \text{prob} \left\{ M_N < (c + \epsilon)N^{1/2} \right\} - \text{prob} \left\{ M_N < cN^{1/2} \right\} \right] \geq 0 \).

The following corollary is a simple consequence of Theorem 1.1:

Corollary 1.1. Let \( N' \) be any positive integral valued and strictly increasing function of \( N \) for which \( \text{prob} \left\{ M_{N'} < cN'^{1/2} \right\} \) converges to a limit function \( P(c) \) at all continuity points \( c \) of \( P(c) \) as \( N \to \infty \). Then also
(1.7) \[ \lim_{N \to \infty} \text{prob} \{ M_{N'} < c N'^{1/2} \} = P(c) \]

at all continuity points \( c \) of \( P(c) \).

The validity of Corollary 1.1 can be derived from that of Theorem 1.1 as follows: Let \( c = c_0 \) be a continuity point of \( P(c) \) and substitute \( N' \) for \( N \) in (1.5) and (1.6). For any positive \( \rho \) all limit points of \( \text{prob} \{ \overline{M}_{N'} < (c_0 - \varepsilon) N'^{1/2} \} \) and \( \text{prob} \{ \overline{M}_{N'} < (c_0 + \varepsilon) N'^{1/2} \} \) will lie in the interval \([P(c_0) - \rho, P(c_0) + \rho]\) for sufficiently small \( \varepsilon \). Hence, equations (1.5) and (1.6) imply that

\[
P(c_0) - \rho \leq \liminf_{N \to \infty} \text{prob} \{ M_{N'} < c_0 N'^{1/2} \}
\]
\[
\leq \limsup_{N \to \infty} \text{prob} \{ M_{N'} < c_0 N'^{1/2} \} \leq P(c_0) + \rho.
\]

Since (1.8) is true for any positive number \( \rho \), Corollary 1.1 is proved.

The result in Corollary 1.1 can be expressed also by saying that for any subsequence \( \{ N' \} \) of \( \{ N \} \) for which \( \overline{M}_{N'}/N'^{1/2} \) has a limiting distribution as \( N \to \infty \), also \( M_{N'}/N'^{1/2} \) has a limiting distribution which is equal to that of \( \overline{M}_{N'}/N'^{1/2} \).

It can easily be verified that the conditions (e) and (f) can always be satisfied for chance variables \( X_{N1}^* \) which take only the values \( 1 \) and \( -1 \) with properly chosen probabilities. Thus, the results of §3 may be used to compute

\[
\text{prob} \left\{ M_N^* < N^{1/2} \left( \frac{\sigma_{N1}^2 + \cdots + \sigma_{NN}^2}{\sigma_{N1}^2 + \cdots + \sigma_{NN}^2} \right)^{1/2} \right\}.
\]

2. Derivation of upper and lower bounds for \( M_N \). Let \( X_1, \cdots, X_N \) be a set of \( N \) variables and let

\[
X_i = X_{N-i+1} \quad (i = 1, 2, \cdots, N).
\]

Let, furthermore,

\[
\tilde{M}_i = \max (X_i, X_i + X_{i-1}, \cdots, X_i + \cdots + X_1),
\]

\( (i = 1, \cdots, N) \).

Clearly

\[
\tilde{M}_N = M_N = \max (X_1, X_1 + X_2, \cdots, X_1 + \cdots + X_N).
\]

If \( X_1, \cdots, X_N \) are independent chance variables, the chance variables \( \tilde{M}_1, \tilde{M}_2, \cdots, \tilde{M}_N \) form a simple Markoff chain, that is, the conditional distribution of \( \tilde{M}_{i+1} \), given \( \tilde{M}_1, \cdots, \tilde{M}_i \), depends only
on $\bar{M}_i$. This is an immediate consequence of the relations:

(2.4) \[ \bar{M}_{i+1} = \bar{M}_i + \bar{X}_{i+1} \quad \text{if } \bar{M}_i > 0 \]

and

(2.5) \[ \bar{M}_{i+1} = \bar{X}_{i+1} \quad \text{if } \bar{M}_i \leq 0. \]

We shall now prove the following theorem:

**Theorem 2.1.** The inequality

(2.6) \[ M_i \leq |\epsilon_1 X_1 + \cdots + \epsilon_i X_i| \quad (i = 1, \ldots, N) \]

holds where $\epsilon_1 = 1$, $\epsilon_i = 1$ if $\epsilon_1 X_1 + \cdots + \epsilon_{i-1} X_{i-1} > 0$ and $\epsilon_i = -1$, if $\epsilon_1 X_1 + \cdots + \epsilon_{i-1} X_{i-1} \leq 0$.

**Proof.** Clearly, (2.6) holds for $i = 1$. We shall prove (2.6) for $i + 1$ assuming that it holds for $i$. For this purpose it is sufficient to show, because of (2.4) and (2.5), that

(2.7) \[ |\epsilon_1 X_1 + \cdots + \epsilon_i X_i| - |\epsilon_1 X_1 + \cdots + \epsilon_i X_i| \geq X_{i+1}. \]

Denote $|\epsilon_1 X_1 + \cdots + \epsilon_i X_i|$ by $\zeta_i$. If $\zeta_i > 0$, then $\epsilon_i + 1$ and inequality (2.7) goes over into

(2.8) \[ |\zeta_i + X_{i+1}| - \zeta_i \geq X_{i+1}, \]

which is obviously true. If $\zeta_i \leq 0$, $\epsilon_{i+1} = -1$ and inequality (2.7) is equivalent with

(2.9) \[ |\zeta_i| + X_{i+1} - |\zeta_i| \geq X_{i+1}, \]

which is obviously true. Hence, Theorem 2.1 is proved.

We shall now prove a theorem giving a lower bound for $\bar{M}_i$.

**Theorem 2.2.** The inequality

(2.10) \[ K_i = |\epsilon_1 X_1 + \cdots + \epsilon_i X_i| - 2 \max_{j \leq i} |X_j| \leq \bar{M}_i \quad (i = 1, \ldots, N) \]

holds where the $\epsilon$'s are defined as in Theorem 2.1.

**Proof.** Theorem 2.2 is obviously true for $i = 1$. We shall assume that it is valid for $i$ and we shall prove it for $i + 1$. It follows from (2.4) and (2.5) that

(2.11) \[ \bar{M}_{i+1} - \bar{M}_i \geq \bar{X}_{i+1}, \]

(2.12) \[ \bar{M}_{i+1} \geq \bar{X}_{i+1}. \]
Hence, to prove (2.10) for \(i+1\) assuming that it is true for \(i\), it is sufficient to show that at least one of the following two inequalities holds:

\[
\begin{align*}
(2.13) & \quad \vec{X}_{i+1} - \vec{K}_i \leq X_{i+1}, \\
(2.14) & \quad \vec{K}_{i+1} \leq X_{i+1}.
\end{align*}
\]

Consider first the case when \(|X_{i+1}| \leq |\epsilon_1 X_1 + \cdots + \epsilon_i X_i|\). In this case (2.13) always holds, as can easily be verified. If \(|X_{i+1}| > |\epsilon_1 X_1 + \cdots + \epsilon_i X_i|\) and \(X_{i+1} \geq 0\), then (2.13) holds again. If \(|X_{i+1}| > |\epsilon_1 X_1 + \cdots + \epsilon_i X_i|\) and \(X_{i+1} < 0\), then \(|\epsilon_1 X_1 + \cdots + \epsilon_i X_i + \epsilon_{i+1} X_{i+1}| \leq |X_{i+1}|\) and, therefore, \(\vec{K}_{i+1} \leq |X_{i+1}| - 2 \max_{i \leq i+1} |X_i| \leq -|X_{i+1}| = X_{i+1}\). Thus, in this case the inequality (2.14) holds. This completes the proof of Theorem 2.2.

Since \(M_N = M_N\), Theorems 2.1 and 2.2 yield the following limits for \(M_N\):

\[
|\epsilon_1 X_1 + \cdots + \epsilon_i X_N| - 2 \max_{i \leq N} |X_i| \leq M_N \leq |\epsilon_1 X_1 + \cdots + \epsilon_i X_N|.
\]

Suppose now that \(X_1, \cdots, X_N\) are chance variables such that the conditional distribution of \(X_i (i = 1, \cdots, N)\) for any given values of \(X_{i+1}, \cdots, X_N\) is symmetric around the origin. Then the probability distribution of \(|\epsilon_1 X_1 + \cdots + \epsilon_N X_N|\) is the same as that of \(|X_1 + \cdots + X_N|\), and the distribution of \(|\epsilon_1 X_1 + \cdots + \epsilon_N X_N| - 2 \max_{i \leq N} |X_i|\) equals that of \(|X_1 + \cdots + X_N - 2 \max_{i \leq N} |X_i|\). It then follows from (2.15) that the following theorem holds:

**Theorem 2.3.** If the conditional distribution of \(X_i (i = 1, 2, \cdots, N)\), for any given value of \(X_{i+1}, \cdots, X_N\) is symmetric around the origin, the inequality

\[
\begin{align*}
\text{prob}\{ |X_1 + \cdots + X_N| < c \} \leq \text{prob}\{ M_N < c \} \\
\leq \text{prob}\{ |X_1 + \cdots + X_N| - 2 \max_{i \leq N} |X_i| < c \}
\end{align*}
\]

holds for any value \(c\).

Inequality (2.15) has also some interesting implications for the asymptotic distribution theory of \(M_N\). In most cases we shall be concerned with the limiting distribution of \(M_N / N^{1/2}\) as \(N \to \infty\) (this is the case discussed in §4). If \((1/N^{1/2}) \max_{i \leq N} |X_i|\) converges stochastically to zero, as will usually be the case, inequality (2.15) implies that the limiting distribution of \(M_N / N^{1/2}\) is the same as that of \((1/N^{1/2}) |\epsilon_1 X_1 + \cdots + \epsilon_N X_N|\).
3. The distribution of \( M_N \) when \( X_i \) can take only the values 1 and \(-1\). Let \( X_1, \ldots, X_N \) be independently distributed chance variables such that \( X_i \) can take only the values 1 and \(-1\). Let \( p_i \) denote the probability that \( X_i = 1 \). The probability that \( X_i = -1 \) is then equal to \( 1 - p_i = q_i \).

Let \( X_i \) and \( \tilde{M}_i \) \((i = 1, \ldots, N)\) be defined by (2.1) and (2.2), respectively. One can easily verify that \( \tilde{M}_i \) can take only the values \(-1, 0, 1, 2, \ldots, i\). Let \( c_{ij} \) denote the probability that \( \tilde{M}_i = j \) for \( j = 1, \ldots, i \), and let \( c_{i0} \) be the probability that \( \tilde{M}_i \leq 0 \). It follows from the definition of the \( \tilde{M}_i \)'s that the following recursion formulas hold:

\[
\begin{align*}
&c_{i+1,0} = q_{i+1}c_{i0} + q_{i+1}c_{i1} \\
&c_{i+1,j} = p_{i+1}c_{i,j-1} + q_{i+1}c_{i,j+1} \\
&\quad (j = 1, 2, \ldots, i + 1)
\end{align*}
\]

Since \( \tilde{M}_N = M_N \), we have

\[
\begin{align*}
&\text{prob} \{ M_N = j \} = c_{Nj} \quad \text{for} \ j = 1, \ldots, N, \\
&\text{prob} \{ M_N \leq 0 \} = c_{N0}.
\end{align*}
\]

We shall now construct \( N \) square matrices \( A_1, \ldots, A_N \), each having \( N + 1 \) rows and \( N + 1 \) columns, such that the first row of the product matrix \( A_1A_2 \cdots A_N \) is equal to \( (c_{N0}, c_{N1}, \ldots, c_{NN}) \). Let \( a_{ij}^k \) denote the element in the \( i \)th row and \( j \)th column of the matrix \( A_k \) \((i, j = 1, \ldots, N + 1; k = 1, \ldots, N)\). We put

\[
\begin{align*}
a_{11}^k &= q_k; & a_{i,i+1}^k &= p_k \\
&\quad (i = 1, 2, \ldots, N); & a_{i,i-1}^k &= q_k \\
&\quad (i = 2, 3, \ldots, N + 1)
\end{align*}
\]

and all other elements \( a_{ij}^k \) equal to zero. It then follows easily from the recursion formulas (3.1) and (3.2) that the first row of the product matrix \( A_1A_2 \cdots A_N \) is equal to \( (c_{N0}, c_{N1}, \ldots, c_{NN}) \). Thus, the first row of the product \( A_1A_2 \cdots A_N \) yields the exact probability distribution of \( M_N \).

Starting with the initial values \( c_{10} = q_1, c_{11} = p_1, c_{ij} = 0 \) for \( j > 1 \), the final values \( c_{N0}, c_{N1}, \ldots, c_{NN} \) can be best computed by repeated application of the recursion formulas (3.1) and (3.2).

4. Proof of Theorem 1.1. Let \( \{ X_N^1 \} \) and \( \{ X_N^2 \} \) be two double sequences of chance variables for which conditions (a)-(f) of Theorem 1.1 are fulfilled. Let \( k \) be a positive integer and \( N_1, \ldots, N_k \) a set of positive integers such that \( N_1 < N_2 < \cdots < N_k = N \). Let, furthermore,
(4.1) \( P_{N,k}(c) = \text{prob} \{ \max (S_{NN}, S_{NN}, \ldots, S_{NN}) < cN^{1/2} \} \).

Because of conditions (b) and (c) of Theorem 1.1, there exist two finite values \( A \) and \( B \) such that \( A \geq N\mu_N^2 \) and \( B \leq \sigma_N^2 \) for all \( N \) and \( i \). Let \( \phi(k) \) be an upper bound of the values

\[
(4.2) \quad \frac{N_1}{N}, \frac{N_2 - N_1}{N}, \ldots, \frac{N_k - N_{k-1}}{N}.
\]

For any positive \( \epsilon \) the following inequality holds:

\[
(4.3) \quad P_{N,k}(c - \epsilon) - \frac{\phi(k)}{\epsilon^2} \left[ B + A\phi(k) \right] \leq P_N(c) \leq P_{N,k}(c),
\]

where \( P_N(c) = \text{prob} \{ M_N < cN^{1/2} \} \). Using a method given by Erdös and Kac [1], the author [2] has proved the above inequality when \( \mu_N = \mu_n \), \( \sigma_N = 1 \) and \( N_i = \lfloor jN/k \rfloor \). To adapt the proof given in [2] to the more general case treated here, it is sufficient to replace the right-hand member of (2.6) in [2] by

\[
(4.4) \quad \frac{(N_{i+1} - N_i)B + (N_{i+1} - N_i)^2 \mu_N^2}{\epsilon^2 N},
\]

where \( \mu_N^2 = \max (\mu_N^2, \ldots, \mu_{NN}) \).

For the purpose of proving Theorem 1.1, we shall choose \( N_i \) to be the smallest positive integer for which

\[
(4.5) \quad \sigma_{N1}^2 + \cdots + \sigma_{NN}^2 \geq \frac{j(\sigma_{N1}^2 + \cdots + \sigma_{NN}^2)}{k}.
\]

Since \( \sigma_{N1}^2 \) has a positive lower bound and a finite upper bound, there exists a positive constant \( h \), independent of \( k \), such that \( h/k \) is an upper bound of the values (4.2). It then follows from (4.3) that

\[
(4.6) \quad P_{N,k}(c - \epsilon) - \frac{1}{\epsilon^2 k} \left( a + b/k \right) \leq P_N(c) \leq P_{N,k}(c)
\]

when \( a \) and \( b \) are positive constants independent of \( N \), \( k \), \( c \) and \( \epsilon \).

Clearly, if Theorem 1.1 is true for the special case when \( \sigma_{N1}^2 + \cdots + \sigma_{NN}^2 = \sigma_{N1}^2 + \cdots + \sigma_{NN}^2 \), it must be true also in the general case. Hence, it is sufficient to prove Theorem 1.1 when \( \sigma_{N1}^2 + \cdots + \sigma_{NN}^2 = \sigma_{N1}^2 + \cdots + \sigma_{NN}^2 \). In what follows we shall therefore restrict ourselves to this special case.

Let \( N^*, P_{N^*k}(c) \), and \( P_N(c) \) have the same meaning with reference to the \( X^* \)'s as \( N \), \( P_{N,k}(c) \), and \( P_N(c) \) with reference to the \( X \)'s. Then we have
(4.7) \[ P_{N,k}(c) - \frac{1}{\epsilon k} (a^* + b^*/k) \leq P_{N,k}(c) \leq P_{N,k}(c), \]

where \( a^* \) and \( b^* \) are positive constants independent of \( N, k, c \) and \( \epsilon \).

Let \( G_{k1}^N, G_{k2}^N, \ldots, G_{kk}^N \) be independently and normally distributed chance variables and let the mean and standard deviation of \( G_{ki}^N \) be equal to the mean and standard deviation of \((k/N)^{1/2}(S_{N,k} - S_{N,k-1})\), respectively. Let, furthermore,

(4.8) \[ Q_{N,k}(c) = \text{prob} \left\{ \max (G_{k1}^N, G_{k1}^N + G_{k2}^N, \ldots, G_{k1}^N + \cdots + G_{kk}^N) < ck^{1/2} \right\}. \]

Clearly, the mean and standard deviation of \( G_{ki}^N \) are bounded functions of \( N, k \) and \( i \). Furthermore, the standard deviation of \( G_{ki}^N \) has a positive lower bound. It then follows from condition (d) and the central limit theorem that

(4.9) \[ \lim_{N \to \infty} [Q_{N,k}(c) - P_{N,k}(c)] = 0. \]

Let \( G_{ki}^N \) and \( Q_{N,k}^*(c) \) have the same meaning with reference to the \( X^* \)'s as \( G_{ki}^N \) and \( Q_{N,k}(c) \) with reference to the \( X \)'s. We then have

(4.10) \[ \lim_{N \to \infty} [Q_{N,k}^*(c) - P_{N,k}^*(c)] = 0. \]

It follows from condition (f) of Theorem 1.1 that

(4.11) \[ \lim_{N \to \infty} E(G_{ki}^N - G_{ki}^N) = 0, \]

(4.12) \[ \lim_{N \to \infty} E(G_{ki}^N - G_{ki}^N)^2 = 0. \]

Hence

(4.13) \[ \lim_{N \to \infty} [Q_{N,k}(c) - Q_{N,k}^*(c)] = 0. \]

From (4.9) and (4.10) and (4.13) we obtain

(4.14) \[ \lim_{N \to \infty} [P_{N,k}(c) - P_{N,k}^*(c)] = 0. \]

Equations (4.6) and (4.14) give

(4.15) \[ \lim_{N \to \infty} \inf \left[ P_{N}(c) - P_{N,k}^*(c - \epsilon) + \frac{1}{\epsilon k} \left( a + \frac{b}{k} \right) \right] \geq 0 \]

and
Since
\[ P_{N, k}(c - \varepsilon) \geq P_{N}(c - \varepsilon) \]
and since, because of (4.7),
\[ P_{N, k}(c) - \frac{1}{\varepsilon^2 k} \left( \frac{a}{k} + \frac{b}{k} \right) \leq P_{N}(c + \varepsilon), \]
we obtain from (4.15) and (4.16)
\[ \lim \inf \left[ P_{N}(c) - P_{N}^{*}(c - \varepsilon) + \frac{1}{\varepsilon^2 k} \left( \frac{a}{k} + \frac{b}{k} \right) \right] \geq 0 \]
and
\[ \lim \inf \left[ P_{N}^{*}(c + \varepsilon) + \frac{1}{\varepsilon^2 k} \left( \frac{a}{k} + \frac{b}{k} \right) - P_{N}(c) \right] \geq 0. \]

Hence, since \( k \) can be chosen arbitrarily large, we obtain
\[ \lim \inf \left[ P_{N}^{*}(c) - P_{N}^{*}(c - \varepsilon) \right] \geq 0 \]
and
\[ \lim \inf \left[ P_{N}^{*}(c + \varepsilon) - P_{N}(c) \right] \geq 0. \]

This concludes the proof of Theorem 1.1. It may be of interest to note that (4.21) and (4.22) imply that for any subsequence \( \{ N' \} \) of the sequence \( \{ N \} \) we have
\[ \lim \inf_{N \to \infty} P_{N'}^{*}(c - \varepsilon) \leq \lim \inf_{N \to \infty} P_{N'}(c) \leq \lim \sup_{N \to \infty} P_{N'}(c) \]
\[ \leq \lim \sup_{N \to \infty} P_{N'}^{*}(c + \varepsilon). \]

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