A NOTE ON LACUNARY POLYNOMIALS

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1. Introduction. In the present note we shall give an elementary derivation of some new bounds for the $p$ smallest (in modulus) zeros of the polynomials of the lacunary type

$$f(z) = a_0 + a_1 z + \cdots + a_p z^p + a_{n_1} z^{n_1} + a_{n_2} z^{n_2} + \cdots + a_{n_k} z^{n_k}, \quad a_0 a_p \neq 0, \quad 0 < p = n_0 < n_1 < \cdots < n_k.$$ 

This will be done by the iterated application, first, of Kakeya's Theorem\(^1\) that, if a polynomial of degree $n$ has $p$ zeros in a circle $C$ of radius $R$, its derivative has at least $p - 1$ zeros in the concentric circle $C'$ of radius $R' = R/(n, p)$; and, secondly, of the specific limits

$$\phi(n, p) \leq \csc \left[ \frac{\pi}{2(n - p + 1)} \right],$$

$$\phi(n, p) \leq \prod_{j=1}^{n-p} \frac{n+j}{n-j}$$

furnished by Marden\(^2\) and Biernacki\(^3\) respectively.

2. Derivation of the bounds. An immediate corollary to Kakeya's Theorem is:

**Theorem I.** If the derivative of an $n$th degree polynomial $P(z)$ has at most $p - 1$ zeros in a circle $\Gamma$ of radius $\rho$, then $P(z)$ has at most $p$ zeros in the concentric circle $\Gamma'$ of radius $\rho' = \rho/\phi(n, p+1)$.

We shall use Theorem I to prove the following theorem.

**Theorem II.** If all the zeros of the polynomial

$$f_0(z) = n_1 n_2 \cdots n_k a_0 + (n_1 - 1)(n_2 - 1) \cdots (n_k - 1)a_1 z$$

$$+ \cdots + (n_1 - p)(n_2 - p) \cdots (n_k - p)a_p z^p$$

lie in the circle $|z| \leq R_0$, at least $p$ zeros of polynomial (1.1) lie in the circle

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For this purpose we define the sequence of polynomials

\begin{align}
F_0(z) &= z^{n_0}f(1/z), \\
F_j(z) &= z^{1-n_k-n_l+j}F_{j-1}(z), \quad j = 1, 2, \ldots, k.
\end{align}

We may verify easily that

\begin{equation}
F_k(z) = z^p f_0(1/z).
\end{equation}

All the zeros of \( F_k(z) \) therefore lie outside the circle \( |z| \geq (1/R_0) \). By equation (2.3), the zeros of \( F_{k-1}^{(j)}(z) \) are the zeros of \( F_k(z) \) and a zero of multiplicity \( n_1 - p - 1 \) at the origin and, hence, only the latter lies inside \( |z| < 1/R_0 \). By Theorem I, \( F_{k-1}(z) \) has at most \( n_1 - p \) zeros in

\[ |z| < [R_0 \phi(n_1, n_1 - p + 1)]^{-1} = 1/R(p, 1). \]

Let us now assume, as already verified for \( j = 1, 2, \ldots, s \), that \( F_{k-j}(z) \) has at most \( n_j - p \) zeros in the circle \( |z| < 1/R(p, j) \). From equations (2.3) with \( j \) replaced by \( k-s \), it follows then that \( F_{k-s}(z) \) has zeros of total multiplicity at most

\[ (n_{s+1} - n_s - 1) + (n_s - p) = n_{s+1} - p - 1 \]

in this circle. By Theorem I, therefore, \( F_{k-s}(z) \) has at most \( n_{s+1} - p \) zeros in the circle

\[ |z| < [R(p, s) \phi(n_{s+1}, n_{s+1} - p + 1)]^{-1} = 1/R(p, s + 1). \]

By mathematical induction, it follows that \( F_0(z) \) has at most \( n_k - p \) zeros in the circle \( |z| < 1/R(p, k) \).

By (2.2), \( f(z) \) has therefore at most \( n_k - p \) zeros outside the circle \( |z| = R(p, k) \) and hence at least \( p \) zeros in or on this circle.

By using the limits (1.2) and (1.3), we now deduce from Theorem II the following corollary.

**Corollary 1.** At least \( p \) zeros of polynomial (1.1) lie in each of the circles

\begin{align}
|z| &\leq R_0 \csc^k \left( \pi/2p \right), \\
|z| &\leq R_0 \prod_{i=1}^{k} \prod_{j=1}^{p-1} (n_i + j)/(n_i - j).
\end{align}

If it is known that all the zeros of the polynomial

\[ h(z) = a_0 + a_1z + \cdots + a_pz^p \]
lie in the circle $|z| \leq R_1$, then the application of a theorem in a previous paper permits us to take

$$R_0 \leq \left[ R_1 n_2 \cdots n_k / (n_1 - p)(n_2 - p) \cdots (n_k - p) \right] = R_2.$$  

As (2.6) with $R_0$ replaced by $R_2$ is the bound furnished recently by Biernacki, we see that the bound (2.6) is at least as good as his bound.

3. **Application to lacunary series.** We shall now use Corollary 1 to prove the following theorem.

**Theorem III.** Let $\rho_1, 0 < \rho_1 \leq \infty$, be the radius of convergence of the series

$$g(z) = a_0 + a_1 z + \cdots + a_p z^p + a_{n_1} z^{n_1} + a_{n_2} z^{n_2} + \cdots,$$

$$a_0 a_p \neq 0, \quad 1 \leq p < n_1 < n_2 < \cdots.$$  

Let the series $\sum (1/n_j)$ be convergent, so that the product

$$A(m) = \prod_{j=1}^{\infty} [1 - (m/n_j)]$$

is also convergent. Let $\rho$, the radius of the circle $|z| = \rho$ containing all the zeros of the polynomial

$$G(z) = A(0) a_0 + A(1) a_1 z + \cdots + A(p) a_p z^p,$$

be such that

$$\rho \prod_{j=1}^{p-1} A(-j)/A(j) = \rho_2 < \rho_1.$$  

Then $g(z)$ has at least $p$ zeros in the circle $|z| \leq \rho_2$.

Let us consider equations (1.1) and (2.1) as defining the sequences of polynomials $f(z, k)$ and $f_0(z, k)$ respectively. When $k \to \infty$, the sequence $[f_0(z, k)/n_1 n_2 \cdots n_k]$ converges uniformly to $G(z)$ in $|z| \leq \rho$. By Hurwitz' theorem, for any given positive $\epsilon$, we may choose a positive $k_1$ so large that all the zeros of each $f_0(z, k)$, $k \geq k_1$, lie in the circle $|z| \leq \rho + \epsilon$. By Corollary 1, at least $p$ zeros of the $f(z, k)$, $k \geq k_1$, lie in the circle

$$|z| \leq (\rho + \epsilon) \prod_{j=1}^{p-1} \prod_{i=1}^{k} (1 + j/n_i)/(1 - j/n_i) < \rho_2 + \epsilon (\rho_2/\rho) = \rho_2'.$$

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Choosing \( \epsilon \) so small that \( \rho'_1 + \epsilon < \rho_1 \), we see that the \( f(z, k) \) converge uniformly to \( g(z) \) in \( |z| \leq \rho'_1 \). Thus \( g(z) \) has \( p \) zeros in the circle \( |z| < \rho'_1 + \epsilon \) and, since \( \epsilon \) is arbitrary, in the circle \( |z| \leq \rho_2 \).

As a corollary to Theorem III, we may prove that, if \( g(z) \) is an entire function, it assumes every finite value an infinite number of times.

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