ON THE FACTORISATION OF ORTHOGONAL TRANSFORMATIONS INTO SYMMETRIES

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Consider the quadratic form

\[ g_{11}(x^1)^2 + g_{22}(x^2)^2 + \cdots + g_{nn}(x^n)^2 \]

where each \( g_{ij} \) is +1 or -1 (but fixed once for all). A homogeneous linear transformation in the variables \( x^1, \ldots, x^n \) (or its matrix) is called orthogonal if it leaves this form invariant. If and only if all \( g_{ij} \) have the same sign we have orthogonality in the classical sense. In any sense it can be easily seen that the orthogonal transformations form a group and that they have determinants \( \pm 1 \).

All quantities considered may be either in the real field or in the complex field (or in any field whatever).

Let \( g_{ij} = 0 \) (\( i \neq j \)). Then, if \( s^i \) is a contravariant vector, its covariant is defined by \( s_i = g_{ik}s^k \) (summation convention). The vector \( s \) is said to be isotropic if it has zero length, that is, if \( s^i s_i = 0 \). If \( s \) is not an isotropic vector, the matrix

\[ \delta_k = \delta_k^i - 2s^i s_k / s_s, \]

where \( \delta_k = 1 \) or 0 if \( i = k \) or \( i \neq k \), defines a special orthogonal transformation of determinant \( -1 \) which we call a symmetry.\(^1\)

Now, by a series of arguments, E. Cartan has proved the following theorem.

**Theorem.**\(^2\) Every orthogonal transformation is decomposable into the product of a number not greater than \( n \) of symmetries.

I shall give a short proof\(^3\) of this theorem below:

Let \( a_k^i \) be the matrix of an arbitrarily given orthogonal transformation. Form the product matrix

\[ b_k^i = s_j a_k^j = a_k^i - 2s^i s_j a_k^j / s_s \]

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3 The first half of the present proof is similar to a part of another simple proof given by W. Givens, *Factorisation and signatures of Lorentz matrices*, Bull. Amer. Math. Soc. vol. 46 (1940) p. 82, but Givens' proof only establishes the weaker statement that the number of symmetries in the factorisation is not greater than \( 2n \).
and consider its first column \( b_1 \). We show that one can choose the vector \( s \) such that \( b_1 = \delta_1 \). In fact, it suffices for this to take \( s^i = \rho (a_1^i - \delta_1^i) \), \( \rho \neq 0 \). Then we find

\[
 s^i s^i_h = g_{hi} s^i = 2 \rho \ g_{i11} (1 - a_1^i).
\]

(1) If \( a_1^i \neq 1 \), we have \( s^i s^i_h \neq 0 \) and so \( s \) is not an isotropic vector. Then

\[
 b_1^i = a_1^i - 2 s^i s^i_h a_1^i / 2 \rho \ g_{i11} (1 - a_1^i) = \delta_1^i.
\]

This being the first column of the matrix \( b_1 \), the orthogonality between this column and each of the other columns implies that the matrix \( b \) has the form

\[
 \begin{pmatrix}
 1 & 0 \\
 0 & c
 \end{pmatrix}
\]

where \( c \) is evidently an orthogonal matrix of order \( n - 1 \). Hence, by induction, the theorem is proved in this case.

(2) If \( a_1^i = 1 \), we can evidently transform the given orthogonal transformation by another orthogonal transformation so as to render \( a_1^i \neq 1 \), and we have only to observe that the symmetries are transformed again into symmetries by an orthogonal transformation.

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