

## ON THE DENSITY OF SOME SEQUENCES OF INTEGERS

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Let  $a_1 < a_2 < \dots$  be any sequence of integers such that no one divides any other, and let  $b_1 < b_2 < \dots$  be the sequence composed of those integers which are divisible by at least one  $a$ . It was once conjectured that the sequence of  $b$ 's necessarily possesses a density. Besicovitch<sup>1</sup> showed that this is not the case. Later Davenport and I<sup>2</sup> showed that the sequence of  $b$ 's always has a logarithmic density, in other words that  $\lim_{n \rightarrow \infty} (1/\log n) \sum_{b_i \leq n} 1/b_i$  exists, and that this logarithmic density is also the lower density of the  $b$ 's.

It is very easy to see that if  $\sum 1/a_i$  converges, then the sequence of  $b$ 's possesses a density. Also it is easy to see that if every pair of  $a$ 's is relatively prime, the density of the  $b$ 's equals  $\prod (1 - 1/a_i)$ , that is, is 0 if and only if  $\sum 1/a_i$  diverges. In the present paper I investigate what weaker conditions will insure that the  $b$ 's have a density. Let  $f(n)$  denote the number of  $a$ 's not exceeding  $n$ . I prove that if  $f(n) < cn/\log n$ , where  $c$  is a constant, then the  $b$ 's have a density. This result is best possible, since we show that if  $\psi(n)$  is any function which tends to infinity with  $n$ , then there exists a sequence  $a_n$  with  $f(n) < n \cdot \psi(n)/\log n$ , for which the density of the  $b$ 's does not exist. The former result will be obtained as a consequence of a slightly more precise theorem. Let  $\phi(n; x; y_1, y_2, \dots, y_n)$  denote generally the number of integers not exceeding  $n$  which are divisible by  $x$  but not divisible by  $y_1, \dots, y_n$ . Then a necessary and sufficient condition for the  $b$ 's to have a density is that

$$(1) \quad \lim_{\epsilon \rightarrow 0} \limsup_{n \rightarrow \infty} \frac{1}{n} \sum_{n^{1-\epsilon} < a_i \leq n} \phi(n; a_i; a_1, a_2, \dots, a_{i-1}) = 0.$$

The condition (1) is certainly satisfied if  $f(n) < cn/\log n$ , since

$$\begin{aligned} \frac{1}{n} \sum_{n^{1-\epsilon} < a_i \leq n} \phi(n; a_i; a_1 \dots a_{i-1}) &< \frac{1}{n} \sum_{n^{1-\epsilon} < a_i \leq n} \left[ \frac{n}{a_i} \right] \\ &< \sum_{n^{1-\epsilon} < m \log m < n} \frac{c'}{m \log m} = O(\epsilon) + O\left(\frac{1}{n}\right). \end{aligned}$$

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<sup>1</sup> Math. Ann. vol. 110 (1934-1935) pp. 336-341.

<sup>2</sup> Acta Arithmetica vol. 2.

As an application of the condition (1) we shall prove that the set of all integers  $m$  which have two divisors  $d_1, d_2$  satisfying  $d_1 < d_2 \leq 2d_1$  exists. I have long conjectured that this density exists, and has value 1, but have still not been able to prove the latter statement.

At the end of the paper I state some unsolved problems connected with the density of a sequence of positive integers.

**THEOREM 1.** *Let  $\psi(n) \rightarrow \infty$  as  $n \rightarrow \infty$ . Then there exists a sequence  $a_1 < a_2 < \dots$  of positive integers such that no one of them divides any other, with  $f(n) < n\psi(n)/\log n$ , and such that the sequence of  $b$ 's does not have a density.*

**PROOF.** We observe first that the condition that one  $a$  does not divide another is inessential here, since we can always select a subsequence having this property, such that every  $a$  is divisible by at least one  $a$  of the subsequence. The condition on  $f(n)$  will remain valid, and the sequence of  $b$ 's will not be affected.

Let  $\epsilon_1, \epsilon_2, \dots$  be a decreasing sequence of positive numbers, tending to 0 sufficiently rapidly, and let  $n_r = n_r(\epsilon_r)$  be a positive integer which we shall suppose later to tend to infinity sufficiently rapidly. We suppose that  $n_r^{1-\epsilon_r} > n_{r-1}$  for all  $r$ . We define the  $a$ 's to consist of all integers in the interval  $(n_r^{1-\epsilon_r}, n_r)$  which have all their prime factors greater than  $n_r^{\epsilon_r}$ , for  $r = 1, 2, \dots$ .

We have first to estimate  $f(m)$ , the number of  $a$ 's not exceeding  $m$ . Let  $r$  be the largest suffix for which  $n_r^{1-\epsilon_r} \leq m$ . If  $m \geq n_r^2$ , then clearly

$$f(m) < n_r \leq m^{1/2} < \frac{m}{\log m}.$$

Suppose, then, that  $m < n_r^2$ . We have

$$f(m) < n_{r-1} + M_\epsilon(m),$$

where  $M_\epsilon(m)$  denotes the number of integers not exceeding  $m$  which have all their prime factors greater than  $m^{\epsilon/2}$ . By Brun's<sup>3</sup> method we obtain

$$M_\epsilon(m) < c_1 m \sum_{p \leq m^{\epsilon/2}} (1 - p^{-1}) < c_2 \frac{m}{\epsilon^2 \log m},$$

where  $c_1, c_2$ , denote positive absolute constants. Hence

$$f(m) < n_{r-1} + c_2 \frac{m}{\epsilon^2 \log m} < \frac{n\psi(m)}{\log m}$$

<sup>3</sup> P. Erdős and M. Kac, Amer. J. Math. vol. 62 (1940) pp. 738-742.

provided  $n_r(\epsilon_r)$  is sufficiently large. It will suffice if

$$\frac{c_2}{\epsilon_r^2} < \frac{1}{2} \psi(n_r^{1-\epsilon_r}).$$

We have now to prove that the sequence of  $b$ 's (the multiples of the  $a$ 's) have no density. Denote by  $A(\epsilon, n)$  the density of the sequence of all integers which have at least one divisor in the interval  $(n^{1-\epsilon}, n)$ . In a previous paper<sup>4</sup> I proved that  $A(\epsilon, n) \rightarrow 0$  if  $\epsilon \rightarrow 0$  and  $n \rightarrow \infty$  independently. Thus if  $\epsilon \rightarrow 0$  and  $n \rightarrow \infty$  sufficiently fast, we have

$$(2) \quad \sum_{r=1}^{\infty} A(\epsilon_r, n_r) < \frac{1}{2}.$$

Denote the number of  $b$ 's not exceeding  $m$  by  $B(m)$ . It follows from (2) that if  $n_r \rightarrow \infty$  sufficiently rapidly, and  $m = n_r^{1-\epsilon_r}$ , then

$$(3) \quad B(m) < m/2.$$

This proves that the lower density of the  $b$ 's is at most  $1/2$ .

Next we show that the upper density of the  $b$ 's is 1, and this will complete the proof of Theorem 1. It suffices to prove that

$$(4) \quad n_r - B(n_r) = o(n_r),$$

in other words that the number of integers up to  $n_r$  which are not divisible by any  $a$  is  $o(n_r)$ . Consider any integer  $t$  satisfying  $n_r^{1-\epsilon_r/2} < t \leq n_r$ , and define

$$(g_{\epsilon_r}(t)) = g_r(t) = \prod_p' p^{\alpha},$$

where the dash indicates that the product is extended over all primes  $p$  with  $p \leq n_r^{\epsilon_r}$ , and  $p^{\alpha}$  is the exact power of  $p$  dividing  $t$ .

If  $g_r(t) < n_r^{\epsilon_r/2}$ , then  $t$  is divisible by an  $a$ , since  $t/g_r(t) > n_r^{1-\epsilon_r}$  and  $t/g_r(t)$  has all its prime factors greater than  $n_r^{\epsilon_r/2}$ , and so is an  $a$ . Hence

$$(5) \quad n_r - B(n_r) < n_r^{1-\epsilon_r/2} + C(n_r),$$

where  $C(n_r)$  denotes the number of integers  $t \leq n_r$  for which  $g_r(t) \geq n_r^{\epsilon_r/2}$ . We recall that the exact power of a prime  $p$  dividing  $N!$  is

$$\sum_{\nu=1}^{\infty} \left[ \frac{N}{p^{\nu}} \right] < \sum_{\nu=1}^{\infty} \frac{N}{p^{\nu}} = \frac{N}{p-1}.$$

Hence

<sup>4</sup> J. London Math. Soc. vol. 11 (1936) pp. 92-96.

$$\prod_{t=1}^{n_r} g_r(t) \leq \prod_{\substack{p \leq n_r \\ p \leq n_r^2}} p^{n_r/p-1} = \exp \left( n_r \sum_{\substack{p \leq n_r \\ p \leq n_r^2}} \frac{\log p}{p-1} \right) < \exp (c_3 \epsilon_r n_r \log n_r) = n_r^{c_3 \epsilon_r^2 n_r}.$$

Hence  $(n_r^{\epsilon_r/2})^{C(n_r)} < n_r^{c_3 \epsilon_r^2 n_r}$ , whence

$$(6) \quad C(n_r) < 2c_3 \epsilon_r n_r.$$

Substituted in (5), this proves (4), provided that  $n_r^{\epsilon_r} \rightarrow \infty$ , which we may suppose to be the case. This completes the proof of Theorem 1.

**THEOREM 2.** *A necessary and sufficient condition that the b's shall have a density is that (1) shall hold.*

**PROOF.** The necessity is easily deduced from an old result. Davenport and I<sup>2</sup> proved that the logarithmic density of the b's exists and has the value

$$\lim_{i \rightarrow \infty} \lim_{n \rightarrow \infty} \frac{1}{n} \sum_{j \leq i} \phi(n; a_j; a_1, \dots, a_{j-1}).$$

Thus if the density of the b's exists, we obtain

$$\lim_{i \rightarrow \infty} \lim_{n \rightarrow \infty} \frac{1}{n} \sum_{j > i} \phi(n; a_j; a_1, \dots, a_{j-1}) = 0.$$

This proves the necessity of (1).

The proof of the sufficiency is much more difficult. We have

$$B(n) = \sum_{a_i \leq n} \phi(n; a_i; a_1, \dots, a_{i-1}) = \sum_1 + \sum_2 + \sum_3,$$

where  $\sum_1$  is extended over  $a_i \leq A$ ,  $\sum_2$  over  $A < a_i \leq n^{1-\epsilon}$ ,  $\sum_3$  over  $n^{1-\epsilon} < a_i \leq n$ . Here  $A = A(n)$  will be chosen later to tend to infinity with  $n$ . By the hypothesis (1) we have

$$(7) \quad \lim_{\epsilon \rightarrow 0} \limsup_{n \rightarrow \infty} \frac{1}{n} \sum_3 = 0.$$

It follows from the earlier work<sup>2</sup> that if  $A = A(n)$  tends to infinity sufficiently slowly, then  $(1/n) \sum_1$  has a limit, this limit being the logarithmic density of the b's, and also

$$\lim_{j \rightarrow \infty} \left( \sum_{i \leq j} \frac{1}{a_i} - \sum_{i_1 < i_2 \leq j} \frac{1}{[a_{i_1}, a_{i_2}]} + \dots \right).$$

Thus the proof of Theorem 2 will be complete if we are able to prove that

$$(8) \quad \frac{1}{n} \sum_2 = \lim_{\epsilon \rightarrow 0} \limsup_{n \rightarrow \infty} \frac{1}{n} \sum_{A < a_i \leq n^{1-\epsilon}} \phi(n; a_i; a_1, \dots, a_{i-1}) = 0.$$

We have

$$\phi(n; a_i; a_1, \dots, a_{i-1}) = \phi\left(\frac{n}{a_i}, 1; d_1^{(i)} \dots\right),$$

where

$$d_j^{(i)} = \frac{a_j}{(a_i, a_j)}.$$

We shall prove that

$$(9) \quad \lim_{\epsilon \rightarrow 0} \limsup_{n \rightarrow \infty} \frac{1}{n} \sum_{A < a_i \leq n^{1-\epsilon}} \phi'\left(\frac{n}{a_i}; 1; d_1^{(i)} \dots\right) = 0$$

where the dash indicates that we retain only those  $d_j^{(i)}$  which satisfy  $d_j^{(i)} < n^\epsilon$ . Clearly (8) follows from (9). (Since  $n^\epsilon \rightarrow \infty$ , not all the  $d_j^{(i)}$  are greater than or equal to  $n^\epsilon$ .)

We define  $g_\epsilon(t)$  as before, with  $n$  in place of  $n_r$  and  $\epsilon$  in place of  $\epsilon_r$ . It follows from (5) and (6) that it will suffice to prove that

$$(10) \quad \lim_{\epsilon \rightarrow 0} \limsup_{n \rightarrow \infty} \frac{1}{n} \sum_{A < a_i \leq n^{1-\epsilon}} \phi''\left(\frac{n}{a_i}; 1; d_1^{(i)} \dots\right) = 0,$$

where  $\phi''(n/a_i; 1; d_1^{(i)} \dots)$  denotes the number of integers  $m$  satisfying

$$(11) \quad m \leq \frac{n}{a_i}; \quad m \not\equiv 0 \pmod{d_j^{(i)}}, \quad d_j^{(i)} < n^\epsilon; \quad g_\epsilon(m) < n^{\epsilon/2}.$$

Consider the integers satisfying (11). They are of the form  $u \cdot v$  where  $u < n^{\epsilon/2}$  and all prime factors of  $u$  are less than  $n^\epsilon$ ,  $u \not\equiv 0 \pmod{d_j^{(i)}}$  for  $d_j^{(i)} < n^\epsilon$ , and all prime factors of  $v$  are greater than  $n^\epsilon$ . We obtain by Brun's method<sup>3</sup> that the number of integers  $m \leq n/a_i$  with fixed  $u$  does not exceed  $(n/u \cdot a_i > n^{\epsilon/2})$

$$(12) \quad c_4 \frac{n}{a_i u} \prod_{p < n^\epsilon} (1 - p^{-1}).$$

Thus the number  $N_i$  of integers satisfying (11) does not exceed

$$(13) \quad c_4 \frac{n}{a_i} \sum' \frac{1}{u} \prod_{p < n^\epsilon} (1 - p^{-1}) \geq \phi''\left(\frac{n}{a_i}; 1; d_1^{(i)} \dots\right),$$

where the dash indicates that the summation is extended over the  $u < n^{\epsilon/2}$ ,  $u \not\equiv 0 \pmod{d_j^{(i)}}$ ,  $d_j^{(i)} < n^{\epsilon^2}$  and all prime factors of  $u$  are less than  $n^{\epsilon^2}$ .

We have to estimate  $\sum N_i$ . Put

$$(14) \quad \lim_{m \rightarrow \infty} \frac{1}{m} \phi\left(\frac{m}{a_i}; 1; d_1^{(i)} \dots\right) = t_i,$$

where in (14) all the  $d_j^{(i)}$  are considered. (It follows from the definition of the  $d_j^{(i)}$  that they are all less than  $n$ . Thus the limit (14) exists.) It follows from our earlier work<sup>2</sup> that

$$(15) \quad \sum_{a_i > A} t_i = o(1).$$

Next we estimate  $t'_i$  where

$$t'_i = \lim_{m \rightarrow \infty} \frac{1}{m} \phi\left(\frac{m}{a_i}; 1; d_i^{(i)}\right), \quad d_i^{(i)} < n^{\epsilon^2}.$$

Here we use the following result of Behrend<sup>5</sup>

$$\begin{aligned} \lim_{n \rightarrow \infty} \frac{1}{n} \phi(n; 1; a_1, \dots, a_i, b_1, \dots, b_i) \\ \cong \lim_{n \rightarrow \infty} \frac{1}{n^2} \phi(n; 1; a_1, \dots, a_i) \cdot \phi(n; 1; b_1, \dots, b_i) \dots \end{aligned}$$

Thus clearly

$$(16) \quad t'_i \cong t_i \left( \lim_{m \rightarrow \infty} \frac{1}{m} \phi(m; 1; x_r) \right)^{-1} = t_i/t'_i',$$

where  $x_i$  runs through the integers from  $n^{\epsilon^2}$  to  $n$ . It follows from the Sieve of Eratosthenes that the density of integers with  $g_\epsilon(m) = k$  equals

$$\frac{1}{k} \prod_{p < n^{\epsilon^2}} (1 - p^{-1}).$$

Thus clearly

$$t'_i' \cong \sum_{k < n^{\epsilon^2}} \frac{1}{k} \prod_{p \leq n} (1 - p^{-1}) > c_5 \epsilon^2$$

or

$$(17) \quad t'_i \leq t_i/c_5 \epsilon^2.$$

<sup>5</sup> Bull. Amer. Math. Soc. vol. 54 (1948) pp. 681-684.

Thus from (15) and (17),

$$(18) \quad \sum_{a_i > A} t_i = o(1).$$

We have by the Sieve of Eratosthenes

$$(19) \quad t_i' = \frac{1}{a_i} \sum' \frac{1}{x} \prod_{p < n^{\epsilon^2}} (1 - p^{-1})$$

where the dash indicates that  $x \not\equiv 0 \pmod{d_j^{(i)}}$   $d_j^{(i)} < n^{\epsilon^2}$  and all prime factors of  $x$  are less than  $n^{\epsilon^2}$ . Comparing (13) and (19) we obtain

$$(20) \quad N_i < c_4 t_i' n.$$

Thus finally from (10) and (18) we obtain  $\sum_{a_i > A} N_i = o(n)$  which proves (10) and completes the proof of Theorem 2.

**THEOREM 3.** *The density of integers having two divisors  $d_1$  and  $d_2$  with  $d_1 < d_2 < 2d_1$  exists.*

**PROOF.** Define a sequence  $a_1, a_2, \dots$  of integers as follows: An integer  $m$  is an  $a$  if  $m$  has two divisors  $d_1$  and  $d_2$  with  $d_1 < d_2 < 2d_1$ , but no divisor of  $m$  has this property. To prove Theorem 3 it will be sufficient to show that the multiples of the  $a$ 's have a density. Thus by Theorem 2 we only have to show that (1) is satisfied. We shall only sketch the proof.

Clearly the  $a$ 's are of the form  $xy$ , where  $x < y < 2x$ . Thus it will be sufficient to show that the number of integers  $m \leq n$  having a divisor in the interval  $(n^{1/2-\epsilon}, n^{1/2})$  is less than  $\eta n$  where  $\eta \rightarrow 0$  as  $\epsilon \rightarrow 0$ . But I proved that the density  $c_{\epsilon, t}$  of integers having a divisor in  $(t, t^{1+\epsilon})$  satisfies

$$\lim_{\epsilon \rightarrow 0} \lim_{t \rightarrow \infty} c_{\epsilon, t} = 0.$$

A similar argument will prove the above result, and so complete the proof of Theorem 3.

It can be shown that the density of integers having two divisors  $d_1$  and  $d_2$  with  $d_1 < d_2 \leq 2d_1$  and either  $d_1$  or  $d_2$  a prime exists and is less than 1. This result is not quite trivial, since if we denote by  $a_1 < a_2 < \dots$  the sequence of those integers having this property and such that no divisor of any  $a$  has this property, then  $\sum 1/a_i$  diverges.

We now state a few unsolved problems.

I. Besicovitch<sup>1</sup> constructed a sequence  $a_1 < a_2 < \dots$  of integers such that no  $a$  divides any other, and the upper density of the  $a$ 's

