THE REMAINDER IN APPROXIMATIONS
BY MOVING AVERAGES

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1. Introduction. Many of the processes of interpolation or smoothing are of the following sort. A function $L(s)$, defined for all real $s$, characterizes the process. Given a function $x(s)$, the function

$$y(t) = \sum_{j=-\infty}^{\infty} x(j)L(t-j)$$

is constructed, when possible; $y(t)$ is thought of as an approximation of $x(t)$. The remainder in the approximation is

$$R[x] = x(t) - y(t).$$

In the conventional processes of smoothing or interpolation, $L(s)$ is a function which vanishes for all $|s|$ sufficiently large. I. J. Schoenberg has recently introduced a class of formulas (1), (2) in which $L(s)$ is an analytic function and the series (1) does not consist of a finite number of terms.

Schoenberg gives an elegant criterion for recognizing cases in which the approximating process is exact for polynomials of degree $n-1$; that is, cases in which $R[x] = 0$, for all $t$, whenever $x(s)$ is a polynomial of degree $n-1$.

In the present paper we obtain an integral representation of such operations $R[x]$ in terms of the $n$th derivative $x^{(n)}(s)$. The representation is precisely of the sort that holds when $R[x]$ is a linear functional on certain spaces of functions $x(s)$ defined on a finite $s$-interval.

2. The integral representation. We shall consider an operation which is more general than (1), (2). Let $g(s, t)$ be a function which, for each number $t$ in a given set $\mathcal{T}$, is of bounded variation in $s$ on each finite $s$-interval. Given any function $x(s)$, put

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1 The author gratefully acknowledges financial support received from the Office of Naval Research, Navy Department.


3 Loc. cit. Theorem 2B, p. 64. Schoenberg’s criterion is valid whether $L(s)$ is a symmetric function or not.

Throughout the present paper “polynomial of degree $k$” is to be understood as “polynomial of proper degree $k$ or less.”
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(3) \( y(t) = \int_{-\infty}^{\infty} x(s) d_\ast g(s, t), \)

and

\[ R[x] = x(t) - y(t), \quad t \in \mathbb{C}. \]

Unless the contrary is stated, *integrals on infinite ranges are to be understood either as Lebesgue-Stieltjes integrals or as improper Lebesgue-Stieltjes integrals*, that is, limits of integrals over finite intervals as the intervals become infinite. Either convention may be adopted, providing that it is consistently held. We shall say that \( R[x] \) exists if \( y(t) \) and \( x(t) \) exist and are finite for each \( t \in \mathbb{C} \).

The integral (3) reduces to the sum (1) if \( g(s, t) \) is, for each \( t \in \mathbb{C} \), constant on each interval \( j < s < j + 1 \) and if \( g(j + 0, t) - g(j - 0, t) = L(t - j), j = \ldots , -2, -1, 0, 1, \ldots \). The name "moving average" is most appropriate to (3) when \( d_\ast g(s - m, t) = d_\ast g(s, t + m) \) for all numbers \( m \) or for all integers \( m \); we do not require that \( g \) satisfy this condition.

Assume that \( R[x] \) exists and vanishes, for all \( t \in \mathbb{C} \), whenever \( x(s) \) is a polynomial of degree \( n - 1 \) (\( n \geq 1 \)). Put

\( \psi_s' = \psi_s'(s) = \begin{cases} 0 & \text{if } s \leq s', \\ \phi(s, s') & \text{if } s > s'. \end{cases} \)

(4)

For each fixed \( s' \), \( R[\psi_s'] \) exists, since \( \psi_s' \) is a truncated polynomial of degree \( n - 1 \). Hence the function \( k(s', t) = R[\psi_s'] \) is defined for all \( s' \) and all \( t \in \mathbb{C} \). An alternative formula for \( k(s', t) \) is the following:

\[ k(s', t) = \begin{cases} \int_{-\infty}^{s'} \phi(s, s') dg(s) & \text{if } s' < t, \\ -\int_{s'}^{\infty} \phi(s, s') dg(s) & \text{if } s' \geq t, \end{cases} \]

(5)

here* and elsewhere \( dg(s) \) is to be understood as an abbreviation for \( d_\ast g(s, t) \). To establish (5), observe that, if \( s' \geq t, \psi_s'(t) = 0 \), and

\[ R[\psi_s'] = \psi_s'(t) - \int_{-\infty}^{\infty} \psi_s'(s) dg(s) = -\int_{s'}^{\infty} \phi(s, s') dg(s), \]

by (4). The other part of (5) is derived similarly, with the use of the

* Whether the value \( s' \) is included or excluded in the range of integration of these integrals is immaterial, since \( \phi(s', s') = 0 \).
additional fact that

\( R[p(s, s')] = 0 = p(t, s') - \int_{-\infty}^{\infty} p(s, s')dg(s) \), \quad t \in \mathbb{C}.

This relation is true because \( p(s, s') \) is a polynomial of degree \( n - 1 \), for each \( s' \).

*Suppose that \( x(s) \) is a function whose derivative of order \( n - 1 \) exists and is absolutely continuous on every finite \( s \)-interval. Put

\[
I = \int_{-\infty}^{\infty} dg(s) \int_{0}^{s} p(s, s')x^{(n)}(s')ds',
\]

\[
R^*[x] = \int_{-\infty}^{\infty} x^{(n)}(s')k(s', t)ds',
\]

**Theorem.** A necessary and sufficient condition that \( R[x] \) and \( R^*[x] \) exist and be equal is that \( I \) exist and that the order of integration in \( I \) be invertible, for all \( t \in \mathbb{C} \).

**Proof.** For brevity put

\( z = p(s, s')x^{(n)}(s') \).

Sufficiency: Since the order of integration in \( I \) is invertible,

\[
I = \int_{-\infty}^{\infty} dg(s) \int_{0}^{s} zds' = -\int_{-\infty}^{0} ds' \int_{-\infty}^{s'} zdg(s)
\]

\[
+ \int_{0}^{\infty} ds' \int_{s'}^{\infty} zdg(s).
\]

As \( x^{(n-1)}(s) \) is absolutely continuous,

\[
x(s) = x(0) + sx'(0) + \cdots + \frac{s^{n-1}x^{(n-1)}(0)}{(n - 1)!} + \int_{0}^{s} zds'.
\]

Since \( R \) vanishes for polynomials of degree \( n - 1 \) and the integral \( I \) exists, \( R[x] \) exists and

\[
R[x] = R \left[ \int_{0}^{s} zds' \right]
\]

\[
= \int_{0}^{t} p(t, s)x^{(n)}(s')ds' - \int_{-\infty}^{\infty} dg(s) \int_{0}^{s} zds'.
\]

Furthermore,
(10) \[ \int_0^t p(t, s') x^{(n)}(s') ds' = \int_0^t ds' \int_{-\infty}^{s'} zdg(s) + \int_0^t ds' \int_{s'}^{\infty} zdg(s). \]

This may be proved as follows. By (6),

\[ p(t, s') = \int_{-\infty}^{s'} p(s, s') dg(s) = \int_{-\infty}^{s'} \int_{-\infty}^{s'} \int_{-\infty}^{s'} p(s, s') dg(s). \]

For fixed \( t \in \mathbb{C} \), each of the last two integrals is a measurable, essentially bounded function of \( s' \) for \( s' \) between 0 and \( t \); hence (10) follows.

By (9), (10) and (7),

\[ R[x] = \int_{-\infty}^{t} ds' \int_{-\infty}^{s'} \int_{-\infty}^{s'} zdg(s) = \int_{-\infty}^{t} ds' \int_{s'}^{\infty} zdg(s) = R^*[x]. \]

The last equality follows from (5). Thus \( R^*[x] \) and \( R[x] \) exist and are equal.

Necessity: Since \( R[x] \) and \( R^*[x] \) exist and are equal, (11) and (9) hold, and \( I \) exists. Furthermore, (11), (9) and (10) imply (7).

This completes the proof of the theorem.

3. Sufficient conditions. Put

\[ M(s', t) = \begin{cases} \int_{s'}^{t} \int_{-\infty}^{s'} p(s, s') \left| \frac{dg(s)}{ds} \right| ds' & \text{if } s' \leq 0, \\ \int_{s'}^{\infty} \int_{-\infty}^{s'} p(s, s') \left| \frac{dg(s)}{ds} \right| ds' & \text{if } s' > 0, \end{cases} \quad t \in \mathbb{C}. \]

If the integral

\[ J = \int_{-\infty}^{\infty} \left| x^{(n)}(s') \right| M(s', t) ds' \]

is finite for all \( t \in \mathbb{C} \), then \( R[x] \) and \( R^*[x] \) exist and are equal, and \( R^*[x] \) exists as a Lebesgue-Stieltjes integral.

PROOF. The double integral corresponding to \( I \) will exist and (7) will hold, by Fubini’s theorem, since the right side of (7) is majorized by \( J \). Hence, by the previous theorem, \( R[x] \) and \( R^*[x] \) exist and are equal.

That \( R^*[x] \) exists as a Lebesgue-Stieltjes integral may be seen as follows. Suppose that \( t \geq 0 \). (\( t < 0 \) is treated similarly.) The integrals

\[ \int_{-\infty}^{0} ds' \int_{-\infty}^{s'} zdg(s), \quad - \int_{-\infty}^{\infty} ds' \int_{s'}^{\infty} zdg(s) \]
are majorized by $J$. Furthermore, by (6),
\begin{equation}
\int_0^t ds' \int_{-\infty}^{s'} zdg(s) = \int_0^t x^{(n)}(s') p(t, s')ds' - \int_0^t ds' \int_{s'}^\infty zdg(s).
\end{equation}

Now the last integral in (13) is majorized by $J$, and the middle integral is on a finite interval. Hence the integrals (12), (13) exist as Lebesgue-Stieltjes integrals. The sum of (13) and the two integrals (12) is precisely $R^*[x]$, by (11).

Note that, by (8), the integral (3) will exist as a Lebesgue-Stieltjes integral, in the present case, if it is true that (3) with $x(s)$ a polynomial of degree $n - 1$ exists as a Lebesgue-Stieltjes integral.

Anyone of the following conditions is sufficient to imply the finiteness of $J$.

(i) For each $t \in T$, $M(s', t)$ is absolutely integrable and $x^{(n)}(s')$ is essentially bounded, on $-\infty < s' < \infty$.

(ii) For each $t \in T$, $M(s', t)$ is essentially bounded and $x^{(n)}(s')$ is absolutely integrable, on $-\infty < s' < \infty$.

(iii) For each $t \in T$, $g(s, t)$ is constant for sufficiently large $s$ and constant for sufficiently small $s$.

In the particular case in which $R[x]$ is of the form (1), (2),
\[
M(s', t) = \begin{cases} 
\sum_{-\infty < j \leq s'} p(s', j) \left| L(t - j) \right| & \text{if } s' \leq 0, \\
\sum_{s' \geq j \geq \infty} p(j, s') \left| L(t - j) \right| & \text{if } s' > 0.
\end{cases}
\]