SOME ANALOGS OF THE GENERALIZED PRINCIPAL AXIS TRANSFORMATION

N. A. WIEGMANN

It is known that two normal matrices can be diagonalized by the same unitary transformation if and only if they commute; this theorem is ordinarily stated for hermitian matrices. Some generalizations of this theorem are known. According to a theorem due to Eckert and Young, if $A$ and $B$ are two $r \times s$ matrices, there are two unitary matrices $U$ and $V$ such that $UA \cdot V = D_1$ and $UB \cdot V = D_2$, $D_1$ and $D_2$ diagonal matrices with real elements, if and only if $AB^*$ and $B^*A$ are hermitian. It is also known that a set of normal matrices $\{A_i\}$ is reducible to diagonal matrices under the same unitary similarity transformation, $UA_i \cdot U^* = D_i$, if and only if $A_iA_j = A_jA_i$ for all $i$ and $j$. (More generally, it is true that a set of matrices $\{A_i\}$ with elements in the complex field and simple elementary divisors is reducible to diagonal matrices under the same similarity transformation if and only if $A_iA_j = A_jA_i$ for all $i$ and $j$.) The following will be shown to hold:

THEOREM. If $\{A_i\}$ is an arbitrary set of nonzero $r \times s$ matrices, there are unitary matrices $U$ and $V$ of orders $r \times r$ and $s \times s$, respectively, such that $UA_i \cdot V = D_i$, $D_i$ diagonal and real, if and only if $A_iA_j = A_jA_i$ and $A_i^*A_i = A_j^*A_j$ for all $i$ and $j$.

If two unitary matrices $U$ and $V$ exist such that $UA_i \cdot V = D_i$, $D_i$ real for all $i$, then $D_iD_i^* = D_i^*D_i = D_i$ where the $D_i$ are $r \times s$ diagonal matrices (that is, the only nonzero elements appear in the $d_{ii}$ position). Therefore, $A_iA_j = A_jA_i$.

Conversely, let the relations $A_i^*A_i = A_j^*A_j$ and $A_iA_j = A_jA_i$ hold for all $i, j$. The proof is by induction.

(1) The theorem is true for a set of matrices of dimension $1 \times s$,

$A_i = [a_i^0, a_i^1, \ldots, a_i^{(s)}]$. For there exist unitary matrices $U$ and $V$ such that $UA_i \cdot V = [d_i^0, 0, \ldots, 0]$ for $d_i^0$ real and greater than 0 since $A_i \neq 0$. For if $UA_i \cdot V = [d_i^0, d_i^1, \ldots, d_i^{(s)}]$, it follows from $A_iA_i = A_iA_i$ that $d_i^{(s)} = d_i^{(s)} = \cdots = d_i^{(0)} = 0$ and since $d_i^0 \cdot d_i^0 = d_i^0 \cdot d_i^0$ and $d_i^0$ is real, $d_i^0 = d_i^0$. In the same way by means of the second of

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the given conditions, the theorem is true for a set of matrices of dimension \( r \times 1 \).

(2) Assume the theorem to be true for a set of matrices of dimension \( k \times l \) for \( k \leq r, l \leq s - 1 \) and for \( k \leq r - 1, l \leq s \). The theorem will be shown to hold for the dimension \( r \times s \) and the induction will then be complete. Let \( \{A_i\} \) be a set of matrices of dimension \( r \times s \) for which the given conditions hold. Let \( U \) and \( V \) be such that

\[
U A_i V = \begin{bmatrix} D & 0_2 \\ 0_3 & 0_4 \end{bmatrix}
\]

where \( U \) and \( V \) are unitary, \( D \) a nonsingular diagonal matrix with real positive diagonal elements, and the submatrices \( 0_2, 0_3, \) and \( 0_4 \) are null matrices or non-existent. If

\[
U A_i V = \begin{bmatrix} G_i & K_i \\ L_i & H_i \end{bmatrix}
\]

it follows from \( A_i A_i^t = A_i A_i^t \) that \( L_i = 0_3 \), and from \( A_i^t A_i = A_i^t A_i \) that \( K_i = 0_2 \). Also, \( D G_i^t = G_i D \) and \( G_i^t D = D G_i \) from both given conditions. Therefore, \( D^2 G_i^t = D G_i D = G_i D^2 \) so \( D^2 G_i = G_i D^2 \). Since \( D \) consists of positive real numbers and since \( G_i \) commutes with \( D^2 \), \( D G_i = G_i D \). Then \( D G_i = G_i D = D G_i \) and since \( D \) is nonsingular, \( G_i = G_i \) for all \( i \). Then from the given relation \( A_i A_i^t = A_i A_i^t \), it follows that \( G_i G_i^t = G_i G_i^t \) or \( G_i G_i = G_i G_i \); therefore the set of hermitian matrices \( \{D, G_i\} \) are all commutative in pairs and, by the generalized principal axis theorem, there exists a unitary matrix \( U_1 \) which diagonalizes all of them. Let \( U_3 = U_1 + I \) be a unitary matrix of the same dimension as \( U \) and \( U_3 = U_3^t + I \) of the same dimension as \( V \). Then,

\[
U_3 U A_i V U_3 = \begin{bmatrix} D & 0_3 \\ 0_3 & 0_4 \end{bmatrix}, \quad U_2 U A_i V U_3 = \begin{bmatrix} D_i & 0_2 \\ 0_3 & H_i \end{bmatrix}
\]

for all \( i \) where the \( H_i \) are either non-existent or of dimension \( k \times l \) where \( k < r \) and \( l < s \). The theorem follows from the induction hypothesis.

It is to be noted that if the set \( \{A_i\} \) are all \( n \times n \) hermitian matrices for which \( A_i A_i^t = A_i A_i^t \) or \( A_i A_i = A_i A_i \) holds, the principal axis transformation for hermitian matrices is obtained and \( V = U^t \).

According to another result due to Eckert and Young,\(^1\) if \( A \) and \( B \) are \( r \times s \) matrices over the complex field, a necessary and sufficient condition that there exist two unitary matrices \( U \) and \( V \) such that \( U A V = D_1 \) and \( U B V = D_2 \), \( D_1 \) and \( D_2 \) diagonal, is that \( A B^t \) and \( B^t A \) be normal. Since this is a generalization of the earlier result, it would
seem reasonable to hope for an extension to a set of matrices \( \{A_i\} \). A simple example shows that this is not the case, however, and the following theorem holds:

**Theorem.** A necessary and sufficient condition that a set of \( n \times n \) matrices \( \{A_i\} \) be brought into diagonal forms by the same unitary \( U, V \) equivalence transformation, \( U A_i V = D_i \), is that the products \( A_i A_j^* \) and \( A_j A_i^* \) be normal for all \( i, j \) and that \( A_k(A_i A_i^*) = (A_i A_i^*) A_k \) for all \( i, j \) and \( k \).

If \( U A_i V = D_i \) for all \( i \), then the given conditions can be easily verified.

Conversely, let \( \{A_i\} \) be a set of matrices for which \( A_i A_j^* \) and \( A_j A_i^* \) are normal and where \( A_k(A_i A_i^*) = (A_i A_i^*) A_k \). The proof is by induction on the order \( n \). The theorem is trivially true if \( n = 1 \). Assume it to be true for order \( k \leq n - 1 \). Now consider a system of order \( n \). There are two possibilities: for all \( i, j \), either \( A_i A_j^* \) is a scaler matrix or there is at least one pair \( i, j \) such that \( A_i A_j^* \) is not a scaler.

1. If for all \( i, j \), \( A_i A_j^* \) is a scaler, \( A_i A_j^* = k_{ij} I \) and since \( A_i A_j^* \) is similar to \( A_i A_i^* \), it is true that:

\[
A_i A_j^* = k_{ij} I = A_j A_i^* \quad \text{for all } i, j.
\]

There are two possibilities: (a) Either all \( A_i = k_i U_i \) where the \( k_i \) are real positive scalers and \( U_i \) are unitary; then all \( A_i \) are normal and from the above, \( A_i A_j = A_j A_i \) since \( A_j = f(A_j^*) \). In this case the principal axis transformation theorem applies for normal matrices so \( V = U^{et} \) and the theorem is true. (b) There is at least one \( A_i \), say \( A_1 \), not of the above form. There exist two unitary \( U, V \) such that \( U A_1 V = D_1 \) is diagonal with real non-negative elements. Also, \( D_1 \) is not scaler for then \( A_1 = U^{et} D_1 V^{et} = D_1 U^{et} V^{et} \); but this contradicts the assumption. Let \( U A_1 V = A_1 \). Then,

\[
(a): \quad U A_1 V V^{et} A_i^{et} U^{et} = U k_{1i} U^{et} = k_{1i} I = D_1 A_i^{et};
\]

\[
V^{et} A_i^{et} U^{et} U A_1 V = V^{et} k_{1i} V = k_{1i} I = A_i^{et} D_1.
\]

Therefore,

\[
D_1 A_i^{et} = A_i^{et} D_1
\]

and

\[
A_j D_1 = D_1 A_j.
\]

Since \( D_1 \) is not scaler, the \( A_j \) are direct sums of matrices of order
$k \leq n - 1$. But for these matrices the given conditions hold and the theorem is true.

(2) If for some $i, j$ the products $A_i A_j^t$ (and consequently $A_j^t A_i$) are not scalar, there exist for this $A_i$ and $A_j$ unitary matrices $U_{ij}$ and $V_{ij}$ such that

$$U_{ij} A_i V_{ij} = D_i, \quad U_{ij} A_j V_{ij} = D_j.$$  

For all $k$, $A_i (A_j A_k A_i) = (A_i A_j A_k) A_i$.

Apply the $U_{ij}, V_{ij}$ and obtain

$$U_{ij} A_k (A_j A_i) V_{ij} = U_{ij} (A_i A_j A_k) A_i V_{ij}$$

so

$$U_{ij} A_k V_{ij} A_j^t U_{ij} V_{ij} A_i V_{ij} = U_{ij} A_i V_{ij} A_j^t U_{ij} V_{ij} A_k V_{ij}$$

so

$$(U_{ij} A_k V_{ij})(D_i D_j) = (D_i D_j)(U_{ij} A_k V_{ij}).$$

Therefore, the matrix $U_{ij} A_k V_{ij}$ commutes with the nonscalar diagonal matrix $D_i D_j$. Since the unitary transformation may be chosen so that like elements of $D_i D_j$ appear together in order, $U_{ij} A_k V_{ij}$ is a direct sum of matrices of order $m < n$ for all $k$. Since the submatrices satisfy these conditions, the theorem is true by induction.

University of Wisconsin